As we were saying last time an inner product on a vector space $V$ induces a natural isomorphism $V \cong V^*$. Let $\langle \cdot, \cdot \rangle$ denote the element of $V^*$ defined by $x \mapsto \langle x, \cdot \rangle$. Let

$$# : V \mapsto \langle \cdot, \cdot \rangle, \quad \#: V \mapsto V^*$$

so that the monogeneity of $\langle \cdot, \cdot \rangle$ is equivalent to $\ker # = \{0\}$.

$$V \leftrightarrow V^* \quad \text{"natural isomorphism"}$$

(finite dim. only.)

These then help to define the "adjoint" of operator $L : V \to V$.

If $L^* : V \to V$ satisfies

$$\langle Lv, w \rangle = \langle v, L^*w \rangle \quad v, w \in V$$

$L^*$ is called an adjoint of $L$. Adjoints are unique in finite dimensional spaces because of the following. First note that $(v \mapsto \langle Lv, w \rangle) \in V^*$ and must then be be equal to $\#(xw)$ for some $xw \in V$. \[ \Rightarrow \#(xw)(v) = \langle \sum \lambda_i v_i, \sum \mu_j w_j \rangle = \langle Lv, w \rangle \] for all $v, w \in V$. \[ \Rightarrow L^*w = xw \text{ is the unique adjoint of } L. \]

Adjoints have a number of nice and easily proved properties

$$L^*|^+ = L \quad (\text{i.e. } + \text{ is an involution})$$

$$1^* = 1 \quad (LM)^* = M^*L^*$$

$L$ has an inverse if and only if $L^*$ has an inverse ...
Complex spectral theorem: $V$ has an orthonormal basis of eigenvectors iff $L$ is normal.

Real spectral theorem: $V$ has an orthonormal basis of eigenvectors iff $L$ is self-adjoint.

ex:

Theorem: Any operator $L$ can be written

$$L = S \left( L^* L \right)^{1/2}$$

for some isometry $S$.

See Arlen for proof.
Application: The Numerical Laplacian

\[ \Delta u = 0 \quad \text{Laplace eq.} \]
\[ \Delta u = f \quad \text{Poisson eq.} \]
\[ 2 \Delta u = \Delta^2 \quad \text{Diffusion eq.} \]
\[ 2\Delta^2 u = \nabla^2 \Delta u \quad \text{Wave eq.} \]
\[ i \hbar \Delta u = -\frac{\hbar^2}{2m} \Delta^2 u + V u \quad \text{Schrödinger eq.} \]

First let's proceed in the continuous case in hand waving fashion.
This can be done precisely but only later in the course.

We think of \( \Delta \) as an operator on some vector space \( \mathcal{C}(U) \)
of smooth functions \( U \to \mathbb{R} \), with inner product

\[ \langle f, g \rangle = \int_U f(x)g(x) \, dx \]

We'll just assume that \( U \in \mathcal{C}(U) \) are chosen so that this is true.

We'll also assume (Green)

\[ \langle \Delta f, g \rangle = \int_U (\Delta f)g = -\int_U (\nabla f \cdot \nabla g) + \int_{\partial U} (\mathbf{n} \cdot \nabla f) g \]

If \( \partial U = \emptyset \), \( \Delta \) is self adjoint, so that if \( y \) is a solution
to Laplace, \( \langle \Delta y, y \rangle = 0 \quad -\langle \nabla y, \nabla y \rangle = 0 \Rightarrow \nabla y = 0 \Rightarrow y = \text{const.} \) are the only solutions.
e.g. Suppose $\Delta p = p$ and $f$ is a solution to Laplace,
\[ <\Delta p, f> = <p, f> = <p, \Delta f> = 0 \Rightarrow \int_U p(x) = 0. \]
e.g. Eigenvalues of $\Delta$ are $\leq 0$. Suppose $\Delta y = 2y$, $y \neq 0$.
\[ <\Delta y, y> = 2 <y, y> = - <\nabla y, \nabla y> \Rightarrow 2 \leq 0. \]
e.g. $\Delta y = 0$, $\Delta y' = 2y'$, eigenvalue, then
\[ <\Delta y, y'> = 2 <y, y'> = 2' <y, y'> \]
\[ \Rightarrow 2 \neq 2' \Rightarrow <y, y'> = 0. \]

Dropping the boundedless assumption, we can get variational results. Suppose that we want $y \in C(U)$ to satisfy $y(x) = F(x)$ on $x \in \partial U$ boundary points. Let
\[ A = \{ y \in C(U) : y(x) = F(x) \text{ for all } x \in \partial U \} \]
We claim that if $y \in A$ minimizes $E(y) = <\nabla y, \nabla y>$, then it is a solution to Laplace with the desired boundary conditions. To see this, let
\[ A_0 = \{ y \in C(U) : y(x) = 0 \text{ for all } x \in \partial U \} \]
Then if $f \in A_0$, $y + a f \in A$.
\[ E(y + af) = E(y) + a^2 E(f) + 2a <\nabla y, \nabla f> \text{ for all } a \in \mathbb{R} \]
\[ \Rightarrow <\nabla y, \nabla f> = 0 \Rightarrow <\Delta y, f> = 0 \text{ for any } f. \] "Weak solution"
\( C(U) \) is some \( \infty \)-dim.
real vector space which we have not precisely defined.

\[ \langle f, g \rangle \equiv \int_U f(x) g(x) \, dx \]

We assumed that this was an inner product.

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

\( \left( \Delta f \right)_{ij} \equiv \sum_{k} \left( f_{i+k,j} + f_{i,j+k} + f_{i,j+k} + f_{i,j-k} - 4 f_{i,j} \right) / a^2 \)

Exact Laplacian

\[ \text{Numerical Laplacian correct to } O(a^2) \]

Let's look at the numerical Laplacian in more detail.
\((\Delta L f)_{ij} = \frac{1}{a^2} \left( f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{ij} \right)\)

Can be written in terms of "shift" operators:

\((C_x f)_{ij} = f_{i+1,j} \quad (C_x^+ f)_{ij} = f_{i-1,j}\)

\((C_y f)_{ij} = f_{i,j+1} \quad (C_y^+ f)_{ij} = f_{i,j-1}\)

Obviously, \(\langle C_x f, C_x f \rangle = \langle f, f \rangle\), i.e. \(C_x, C_x^+, C_y, C_y^+\) are isometries.

\[\Rightarrow \quad \Delta_L = \frac{1}{a^2} \left( C_x + C_x^+ + C_y + C_y^+ - 4I \right)\]

is self adjoint.

We can also define \(D_x = (C_x - I)/a\), \(D_y = (C_y - I)/a\) so that

\[-\Delta_L = D_x D_x^+ + D_y D_y^+\]

Now, the whole set of results that we suggested but did not prove in the continuous case follows precisely in the lattice case.
### Continuous

\[ U \subset \mathbb{R}^2 \]

\( C(U) \) is an \( n \)-\( \text{dim. real vector space} \)

\[
\langle f, g \rangle = \int U f(u)g(u) \, du
\]

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

\[
\begin{align*}
\Delta f &= \langle f, \Delta g \rangle \\
\langle \Delta f, g \rangle &= -\langle \nabla f, \nabla g \rangle \\
\Delta g &= O \Leftrightarrow \nabla g = O
\end{align*}
\]

\[
E(y) = \langle \nabla y, \nabla y \rangle
\]

### Lattice

\[ U = \{0, 1, 2, \ldots, m-1 \} \times \{0, 1, \ldots, n-1 \} \]

\( C(U) \) is the \( m \times n \) \( \text{dim. free real vector space on } U \)

\[
\langle f, g \rangle = \sum_{u \in U} f_u \cdot g_u
\]

\[
\Delta_L = D_x D_x^T + D_y D_y^T
\]

\[
\begin{align*}
\Delta_L f, g &= \langle f, \Delta_L g \rangle \\
\langle \Delta_L f, g \rangle &= -\langle D_x f, D_x g \rangle - \langle D_y f, D_y g \rangle
\end{align*}
\]

\[
\Delta_L y = O \Leftrightarrow D_x y = D_y y = O
\]

\[
\Delta_L y = O \Rightarrow \langle y, y \rangle = O
\]

\[
E_L(y) = \langle D_x y, D_x y \rangle + \langle D_y y, D_y y \rangle
\]

---

Notice that if we had a basis of eigenvectors of \( C_x, C_y \), this would also be a basis of \( D_x, D_y, D_x^T, D_y^T \) and \( \Delta \).

Let's find this, but first, switch to \( U = \{0, 1, 2, \ldots, m-1 \} \times \{0, 1, \ldots, n-1 \} \)

\( m = \text{even for convenience} \ldots \)
Since shift is an isometry the only possible eigenvalues are $\pm 1$ (because $\langle c y, c y \rangle = \langle y, y \rangle = \lambda^2 \langle y, y \rangle$), and all eigenvectors are proportional to

$$ C \begin{pmatrix} 1 \\ ... \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ ... \\ 1 \end{pmatrix} \quad \text{or} \quad C \begin{pmatrix} 1 \\ -2 \\ ... \\ 1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ -2 \\ ... \\ 1 \end{pmatrix} $$

and these two certainly don't span $\mathbb{C}^n$ in general.

But don't give up! We can try solving these problems in the free complex vector space on $U$ instead. Define $C$ as a shift as before, $D = \frac{1}{2} (C - I)$, $\Delta = -DD^*$ as before.

$$ \langle f, g \rangle = \sum_{u \in U} f_u g_u^* $$

is the new inner product.

$C$ is still an isometry $\Rightarrow \lambda = e^{i \theta_k}$. Since $C^n = 1$, obviously $\lambda^n = 1$ and the only possible eigenvalues are

$$ \lambda \in \{ 2^0, 2^1, 2^2, \ldots, 2^{n-1} \} \quad z = e^{2\pi i/n} $$

$$ C \begin{pmatrix} 2^0 \\ 2^1 \\ ... \\ 2^{n-1} \end{pmatrix} = z \begin{pmatrix} 2^0 \\ 2^1 \\ ... \\ 2^{n-1} \end{pmatrix} \quad C \begin{pmatrix} 2^0 \\ 2^1 \\ ... \\ 2^{n-1} \end{pmatrix} = z^2 \begin{pmatrix} 2^0 \\ 2^1 \\ ... \\ 2^{n-1} \end{pmatrix} \quad \cdots \quad C \begin{pmatrix} 2^0 \\ 2^1 \\ ... \\ 2^{n-1} \end{pmatrix} = z^n \begin{pmatrix} 2^0 \\ 2^1 \\ ... \\ 2^{n-1} \end{pmatrix} $$

$\Rightarrow$ we found eigenvectors for all $n$ eigenvalues $\Rightarrow$ this is a basis making the operator of $C, D, \Delta$ trivial.

Changing to this basis is called the "Fermi Transform."

[There is a very nice $\Theta(n \log n)$ FFT algorithm for this].
A functor from category $C$ to category $C'$ is a collection of mappings which preserve identities and commuting triangles:

\[
\begin{align*}
C & \xrightarrow{\mathcal{F}} \quad X \xrightarrow{f} Y \xrightarrow{g} Z \\
& \downarrow \quad \downarrow \quad \downarrow \\
C' & \quad \mathcal{F}(X) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(Z) \\
& \quad \mathcal{F}(g \circ f)
\end{align*}
\]

For example:

\[
\begin{align*}
\text{Sets} & \quad X \xrightarrow{f} Y \xrightarrow{g} Z \\
& \downarrow \quad \downarrow \quad \downarrow \\
\text{Sets} & \quad \mathcal{P}(X) \xrightarrow{\mathcal{P}(f)} \mathcal{P}(Y) \xrightarrow{\mathcal{P}(g)} \mathcal{P}(Z)
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}(X) & = \{ \text{the set of subsets of } X \} \\
\mathcal{P}(f) : A \mapsto \{ f(a) : a \in A \} \in \mathcal{P}(Y) \\
\mathcal{P}(g \circ f) & = \mathcal{P}(g) \circ \mathcal{P}(f) \\
\mathcal{P}(i_X) & = i_{\mathcal{P}(X)}
\end{align*}
\]

This is a functor.
Composition of functors is defined by composing all the corresponding mappings.

- Composition of functors is associative.

- For each category $C$, there is an identity functor (where all the mappings are identities) which obeys $i_C \circ F = F$, $F \circ i_C = F$.

- The categories of all categories is a category.

You can also easily verify that

Functors preserve isomorphisms.
Free constructions are all functors.

\[ S \xrightarrow{f} S' \xrightarrow{\mathcal{F}} \mathcal{F}(S) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(S') \]

\[ S \cong S' \Rightarrow \mathcal{F}(S) \cong \mathcal{F}(S') \]

Vector spaces with isomorphic bases are isomorphic.
Free constructions actually involve two functors

\[ S \rightarrow V \]

\[ \text{Free: Sets} \Rightarrow V.S.s \]
\[ \text{Furget: V.S.s} \Rightarrow \text{Sets} \]

\[ \text{Mor} \ (\text{Free} S, W) \cong \text{Mor} \ (S, \text{Furget} W) \]

These are called "adjoint" functors.
**Associative Algebra**

An associative algebra is just a vector space $V$ with a bilinear product $V \times V \to V$.

**Examples:**
- Real valued functions on a set with pointwise multiplication.
- $\mathbb{R}^3$ with the usual cross product.
- $\text{End}(V)$ as a vector space with composition as multiplication.

**Morphisms:** Linear maps $V \xrightarrow{\gamma} W$ with $\gamma(uv') = \gamma(u)\gamma(v')$.

$\text{Ker } \gamma = \{ v : \gamma(v) = 0 \}$

$\text{Im } \gamma = \{ \gamma(v) : v \in V \}$

are both subalgebras.

Suppose $A$ is a subalgebra of $V$

- $\text{cosets } u + A$ cover $V$ without overlapping
- $\text{cosets are isomorphic as sets}$
- $\text{try to make the cosets into an } A.A.$

$(u + A) + (u' + A) \mapsto u + u' + A$

$(u + A) \cdot (u' + A) \mapsto uu' + A$

This works if $A$ is an Ideal $\iff$

def: if $v \cdot a \in A \text{ for all } v \in V, a \in A$,

$\forall a, a' \in A$

$uv' = wu' + wa' + aa'$
example \( V = \mathbb{R} \) with pointwise multiplication.

- The subset of bounded functions in \( V \) is a subalgebra but not an ideal because a bounded function multiplied by a non-bounded function may not be bounded.

- The subset of functions in \( V \) which are zero except on \( [0,1] \) is an ideal.

As with groups and normal subgroups, \( \ker y \) is an ideal in the category of real or complex \( a.a.s. \).

\[
\begin{array}{c}
\bigoplus & \bigoplus & \bigoplus & \cdots \leftarrow V^0 \bigoplus V^1 \bigoplus V^2 \bigoplus \cdots \\
\bigoplus & \bigoplus & \bigoplus & \cdots \rightarrow V^0 \bigoplus V^1 \bigoplus V^2 \bigoplus \cdots \\
\downarrow & \downarrow & \downarrow & \cdots \rightarrow V^0 \bigoplus V^1 \bigoplus V^2 \bigoplus \cdots \\
V & \rightarrow V/E \ker y & \rightarrow \text{Im} y & \rightarrow W
\end{array}
\]

\( y \) commutes with natural mappings as usual.

Free associative algebra on \( V \) is \( V^0 \bigoplus V^1 \bigoplus V^2 \bigoplus \cdots \)

\( S^0 \bigoplus S^1 \bigoplus S^2 \bigoplus \cdots \bigoplus \cdots \rightarrow V^0 \bigoplus V^1 \bigoplus V^2 \bigoplus \cdots \)

This is much less scary than it looks. \( S^0 \bigoplus S^1 \bigoplus \cdots \) is just the set of (possibly empty) lists of elements from the set \( S \) and \( V^0 \bigoplus V^1 \bigoplus \cdots \) are linear combinations of the same, with concatenation in multiplication.
A topological space is a set $X$ with a collection of "open" subsets satisfying:

(a) $\emptyset$ and $X$ are both open.
(b) If $U_i$ are open, $\bigcup U_i$ is open.
(c) If $U, V$ are open, $U \cap V$ is open.

Examples:

- Let $X$ be any set. Let open sets be all subsets of $X$. This is the discrete topology on $X$.

- Let $X$ be any set. Let only $\emptyset$ and $X$ be open sets. This is the indiscrete topology on $X$.

- Let $X$ be a metric space and define $B_x^\epsilon = \{y \in X : d(y, x) < \epsilon\}$. Let a subset $\mathcal{O}$ of $X$ be open if, for each $x \in \mathcal{O}$, there is a $B_x^\epsilon \subseteq \mathcal{O}$ with $\epsilon > 0$. This is a topology because:

  (a) Every $x \in \emptyset$ vacuously satisfies the condition.
  (b) $B_x^\epsilon \subseteq X$ for any $\epsilon > 0 \Rightarrow X$ is open.
  (c) If $U_i$ are all open and $x \in \bigcup U_i$, $x \in U_i$ for some $i \Rightarrow B_x^\epsilon \subseteq (\bigcup U_i \subseteq U_i)$ for some $i \Rightarrow \epsilon > 0$.
  (d) If $\mathcal{O}, \mathcal{O}'$ are open, let $x \in \mathcal{O}, \mathcal{O}'$. Let $B_x^\epsilon \subseteq \mathcal{O}$, $B_x^\epsilon' \subseteq \mathcal{O}'$. Let $B_{\min(\epsilon, \epsilon')}^\epsilon \subseteq \mathcal{O} \cap \mathcal{O}' \Rightarrow \mathcal{O} \cap \mathcal{O}'$ is open.

This is the standard topology on $\mathbb{R}^n$. 

-15-
Def: A set is called "closed" if it is the complement of an open set.

⇒ (a) ∅ and X are closed
(b) If $C_2$ are closed, so is $A \cap C_1$
(c) If $C$ and $C'$ are closed, $C \cup C'$ is closed.

Note: Sets may be open, closed, both, or neither.

Characterizing open sets

Theorem: A subset $A$ of a topological space $X$ is open iff every $a \in A$ has an open neighborhood $O_A$.

Proof: Suppose $A$ is open. Then $A \cup A$ is certainly an open neighborhood of any $a \in A$. Conversely, suppose that the property holds, then $A = \bigcup\{O_a \mid a \in A\}$ is open.

Def: Let $A$ be a subset of a topological space $X$.

$\text{Int}(A) \equiv \text{The union of all open subsets of } A$

$\text{CL}(A) \equiv \text{The intersection of all closed subsets of } A$

$\text{Bdry}(A) \equiv \text{CL}(A) - \text{Int}(A)$

Theorem: $A$ is open iff it contains no boundary points.

Theorem: Single points in a Hausdorff space are closed.

Proof. $\square$ $\{x \cap y \} = \bigcup\{y \cap y \}$ is open
where $x \cap y$, by Hausdorffness.