A vector space over a field $F$ is an abelian group $V$ with biadditive $F \times V \to V$ satisfying $a \cdot (b \cdot v) = (ab) \cdot v$ and $1 \cdot v = v$.

Ex: $\mathbb{R}^n$, $\mathbb{C}^n$, Monset $(A, R)$, $n$-degree polynomials over $\mathbb{R}$, ...

In practice, $F$ is almost always $\mathbb{R}$ or $\mathbb{C}$. For the following discussion, choose a fixed $F$.

Let a morphism from $F$-linear space $V$ to $F$-linear space $W$ be a function $g: V \to W$ which is both a group homomorphism

$$g(v + v') = g(v) + g(v')$$

and a "morphism of scalar multiplication"

$$g(a \cdot v) = a \cdot g(v)$$

These are called linear mappings. You can easily check that identity mappings and any composition of linear maps is another linear map $\Rightarrow$ we have the category of $F$-linear spaces for each $F$.

$$g \in \text{Mor}(V, W)$$

are sometimes called "operators".
Just as with groups, good thing happen with the right choice of morphism.

\[ \ker y = \{ v \in V : y(v) = 0 \} \quad \text{Im } y = \{ y(v) : v \in V \} \]

are both vector spaces, \( y(0) = 0 \) and \( y(-v) = -y(v) \), just as with groups.

You can also easily verify

\[
\begin{align*}
\text{\( y \) is monic} & \iff \text{\( y \) is 1-1} & \iff \ker y = \{ 0 \} \\
\text{\( y \) is epic} & \iff \text{\( y \) is onto} & \iff \text{Im } y = W \\
\text{\( y \) is isom} & \iff \text{\( y \) is 1-1 and onto} & \iff \ker y = \{ 0 \} \text{ and } \text{Im } y = W
\end{align*}
\]

(see homework).

This is all the same as in groups.
**Products and Sums**

\[ V \xrightarrow{\alpha} V \times W \xrightarrow{\beta} W \]

\[ V \times W = \text{cartesian product with } (v, w) + (v', w') = (v+v', w+w') \]
\[ a \cdot (v, w) = (av, aw) \]

\[ V \times W \] is, thus, a vector space and the usual

\[ \gamma : u \mapsto (\gamma(u), \psi(u)) \]

is a linear mapping for

\[ \gamma(u + u') = (\gamma(u + u'), \psi(u + u')) = (\gamma(u) + \gamma(u'), \psi(u) + \psi(u')) \]
\[ = (\gamma(u), \psi(u)) + (\gamma(u'), \psi(u')) = \gamma(u) + \gamma(u') \]

and \[ \gamma(a \cdot u) = (\gamma(a \cdot u), \psi(a \cdot u)) = (a \cdot \gamma(u), a \cdot \psi(u)) = a \cdot (\gamma(u), \psi(u)) \]
\[ = a \cdot \gamma(u), \]

The same vector space \[ V \times W \] is actually the direct sum also

\[ V \xrightarrow{\alpha} V \oplus W \xleftarrow{\beta} W \]

\[ \phi : u \mapsto (u, 0) \]
\[ \psi : w \mapsto (0, w) \]
\[ \gamma : (u, w) \mapsto \psi(u) + \psi(w) \]

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Unlike the case of groups, all vector spaces can be created by free construction. Begin with

**Theorem:** A free vector space exists on any set.

**Proof:** Let \( S \) be a set and let

\[
V = \{ \text{\( F \)-valued functions} \; S \rightarrow F \; \text{which are zero except at a finite number of points} \; \}
\]

\( V \) is a vector space in the obvious way with pointwise addition and scalar multiplication: \( f, g \in V \), \( (f + g)(s) = f(s) + g(s) \), \( (a \cdot f)(s) = a \cdot f(s) \).

\[
\begin{array}{ccc}
S & \xrightarrow{\varphi} & V \\
\downarrow \varphi & & \downarrow f \\
S & \xrightarrow{\varphi} & V
\end{array}
\]

Let \( \xi(s) = t \rightarrow \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise} \end{cases} \in V \)

The diagram commuting implies \( f(\xi(s)) = g(s) \) and since any \( \nu \in V \) is a finite linear combination of \( \xi(s) \), we must have

\[
\nu \left( a, \xi(s_1) + a_2 \xi(s_2) + \ldots + a_n \xi(s_n) \right) \\
= a, \nu(s_1) + a_2 \nu(s_2) + \ldots + a_n \nu(s_n).
\]

Note that this defines a linear function \( \varphi : (S, V) \) is the free vector space on \( S \).
Independence / Span / Basis

Given a subset $S$ of $V$, we can form "linear combinations" such as

$$a_1 S_1 + a_2 S_2 + \ldots + a_n S_n$$

If every element of $V$ is such a combination, we say that $S$ spans $V$. If

$$a_1 S_1 + a_2 S_2 + \ldots + a_n S_n = 0$$

$\Rightarrow a_i = 0$, we say that $S$ is independent. If $S \subseteq V$ is independent and spans $V$, we say that $S$ is a basis.

These concepts also nicely characterize monic / epic:

**Fact:** $V \rightarrow W$ is monic iff it preserves independence, i.e. $\gamma[S]$ is independent in $W$ for every independent $S \subseteq V$.

**Fact:** $V \rightarrow W$ is epic iff it preserves span, i.e. $\gamma[S]$ spans $W$ for every $S \subseteq V$ which spans $V$.

**Fact:** $V \rightarrow W$ is iso iff $\gamma$ preserves $\exists$ bases, i.e. if $\gamma[S]$ is a basis for every basis $S$ of $V$.

Proofs: Easy.
**Theorem:** Every vector space has a basis.

**Proof:** Let \( V \) be a vector space and consider the collection of independent subsets of \( V \) partially ordered by inclusion. If \( A_\lambda, \lambda \in \Lambda \) is any totally ordered subset of these, then \( \bigcup \lambda A_\lambda \) is also independent and is an upper bound. By Zerm, there exist maximal independent subsets. Let \( B \) be one of these and suppose that \( B \) does not span \( V \). However, if \( v \) is outside the span of \( B \), \( B \cup \{v\} \) is independent. \( \Rightarrow \Rightarrow B \) is a basis.

**Theorem:** Bases of a vector space are isomorphic as sets.

**Proof:** Follow theorem 20.

\[ \Rightarrow \] Every vector space is the free vector space on any of its sets of basis vectors.

\[ \Rightarrow \] Every vector space is the free vector space on some set.

\[ \Rightarrow \text{def: The dimension of a vector space is defined to be the size of one of its bases. If the bases are finite, then the space is said to be finite-dimensional.} \]
Subspaces of Vector Spaces

Just as with groups, we define a subspace of a vector space to be a subset which is a vector space in its own. Just as with groups, we can define...

Subspace \( U \) of \( V \)
- cosets \( v + U \)
  - cover \( V \) without overlapping
  - are isomorphic as sets
  - form a vector space \( \frac{V}{U} \) with

\[
\forall (v + U) + (v' + U) = (v + v') + U
\]

Also, just as with groups, any morphism \( V \to W \) can be expanded

\[
\begin{array}{c}
V \\ \quad \searrow \quad \searrow \\
\quad \quad \quad epi \quad \quad \quad \quad isom \\
\downarrow \quad \downarrow \quad \downarrow \\
\frac{V}{Ker \gamma} \quad \to \quad \text{Im} \gamma \quad \to \quad W \\
\gamma \\
\downarrow \quad \downarrow \\
U \quad \to \quad U + Ker \gamma \\
\quad \quad \quad \cong \\
\downarrow \quad \downarrow \\
\gamma(U) \quad \to \quad \gamma(U + W) \quad \to \quad \gamma(W)
\end{array}
\]

In vector spaces, we say that subspaces \( U, W \) of \( V \) are complementary if

(a) Every \( v \) can be written \( v = u + w \) for some \( u \in U, w \in W \).

(b) \( u + w = 0 \Rightarrow u = w = 0 \).
Fact: If \( U \) and \( W \) are complementary in \( V \), then \( V = U \oplus W \).

Proof: Conditions (a) and (b) above are precisely that the map 
\[ \Psi : (u, w) \mapsto u + w \] 
has \( \text{Im} \Psi = V \), \( \ker \Psi = \{0\} \).

Theorem: Every subspace of a vector space has a complementary subspace.

Proof: Note that (6) above is equivalent to \( U \cap W = \{0\} \).

Consider subspaces \( A \) of \( V \) with the property \( U \cap A = \{0\} \), partially ordered by inclusion.

If \( W_t \) is any totally ordered subset of these, \( U \cap (\bigcup_t W_t) = \{0\} \)
and \( \bigcup_t W_t \) is a subspace of \( V \) above. By Zorn's, there is a
maximal subspace \( W \) with \( U \cap W = \{0\} \). Suppose, now,
that some \( v \in V \) can't be expressed as \( v = u + w \), \( u \in U, w \in W \).
But then \( U \cap [W \cup W_v] = \{0\} \) because if some element
\( w + a_v = v \Rightarrow a = 0 \Rightarrow w = u = 0 \). This, however, violates the
maximality of \( W \in [W \cup W_v] \Rightarrow W \) is complementary to \( U \).

Example: \( U \) is a plane
through \( \psi \in \mathbb{R}^3 \)
\[ U = \mathbb{R}^3 \]
\[ V = \mathbb{R}^3 \]
any 1-d subspace \( W \)
not in the plane is
complementary.

Example: \( V = \{ \text{real functions on} \ [0,1] \} \)
\[ U = \{ \text{real functions with} \ f(x) = 0 \]
in \( x \in [0, \frac{1}{2}] \} \]
\[ W = \{ \text{real functions with} \ f(x) = 0 \]
in \( x \in (\frac{1}{2}, 1] \} \]
\[ U \text{ and } W \text{ are complementary subspaces of } V. \]
Duality

Given a vector space \( V \) over \( F \), let

\[ V^* = \{ \text{linear maps from } V \text{ to } F \} \]

where, for \( f, g \in V \), \( (f + g)(v) = f(v) + g(v) \) and \( (\alpha f)(v) = \alpha f(v) \) makes \( V^* \) into a vector space also.

**Example 1:** Let \( V = \mathbb{R}^3 \),

\[
\begin{align*}
    dx : (x, y, z) &\mapsto x \\
    dy : (x, y, z) &\mapsto y \\
    dz : (x, y, z) &\mapsto z \\
\end{align*}
\]

In fact, \( \{dx, dy, dz\} \) is a basis of \((\mathbb{R}^3)^*)\).

**Example 2:** Let \( V \) be the vector space of continuous real valued functions on \([0, 1]\). Fix \( p \in V \), then

\[
\begin{align*}
    f &\mapsto \frac{1}{1} \int_0^1 p(x) f(x) \, dx \\
\end{align*}
\]

is an element of \( V^* \).

For every \( V \xrightarrow{y} U \), there is a naturally induced \( V \xleftarrow{y^*} U^* \) defined by

\[
\begin{CD}
    V @>y>> U \\
    \downarrow @>y^*>> \downarrow f \\
    F @>f \circ y^*>> F
\end{CD}
\]

\[ g^* : U^* \rightarrow V^* \]

\[ g^* : f \mapsto f \circ y \]
You can also easily verify that "*" interacts nicely with composition

\[
\begin{array}{c}
V \xrightarrow{\psi} U \xrightarrow{\psi} W \\
\downarrow \quad \downarrow \quad \downarrow \\
F \quad F \quad F
\end{array}
\]

\((\psi \circ \phi)^* = \phi^* \circ \psi^*
\]

Note that \(R^3 \cong (R^3)^*\) in example 1. Is this true in general?

It is almost obvious that \(V \neq V^*\) iff \(V\) is finite dimensional if you notice that for free \(S \to V\) (guaranteed to exist),

\[
\begin{array}{c}
S \xrightarrow{\xi} V \\
\downarrow \quad \downarrow \, \xi \\
S \quad F
\end{array}
\]

\[\xi \mapsto \xi \text{ is a set isomorphism}
\]

so

\[V \cong \{\text{Functions from } S \text{ to } F \text{ which are zero except at a finite number of points}\}
\]

\[V^* \cong \{\text{Functions from } S \text{ to } F \text{ in general}\}
\]

i.e. \(V^*\) is "bigger" than \(V\). If we knew that \(V^*\) can't be isomorphic with a proper subset like \(V\), then we would be done. This, however, is false.

(see problems).
example: Let \( V = \{ \text{continuous real functions on } [0,1] \} \)

\[
\Psi (p) = \left( f \mapsto \int_0^1 p(x) f(x) \, dx \right) \quad \Psi : V \to V^*
\]
as in example 2. \( \Psi \) is 1-1, however, there are elements in \( V^* \) which are not in the image of \( \Psi \). For example

\[
S_y : f \mapsto f(y)
\]
is certainly in \( V^* \) and can't be expressed as above.
These are the "Dirac delta functions".

example: In Dirac's notation, \( |x> \) is a state vector and \( <x| \) is a dual vector. You have to be careful about this because this implies \( V = V^* \) which is false if \( V \) is infinite dimensional.
Multilinear maps and Tensors

In vector space applications one is often dealing with bilinear functions rather than morphisms, e.g.

\[
\text{dot}(v, w) = v_x w_x + v_y w_y + v_z w_z \quad \text{in} \quad \mathbb{R}^3
\]

\[
g(v, w) = v_x w_x + v_y w_y + v_z w_z - v_z w_x \quad \text{in} \quad \mathbb{R}^4
\]

\[
\text{area}(v, w) \rightarrow \text{area in the dotted line}
\]

\[
\text{cov}(v, w) = \frac{1}{2} v^T C w \quad \text{"covariance" in probability.}
\]

These are all maps having the property

\[
M(v + av', w) = M(v, w) + a \cdot M(v', w)
\]

\[
M(v, w + bw') = M(v, w) + b \cdot M(v, w')
\]

\[
M : V \times W \rightarrow \mathbb{Z} \quad V, W, \mathbb{Z} \text{ vector spaces}
\]

\[
\mathbb{Z} \text{ cartesian product only, not direct product.}
\]

Of course, \(V \times W\) is a set and we can always do

\[
V \times W \rightarrow F \quad \text{where} \quad V \times W \rightarrow F \text{ is the free (no dimensioned)}
\]

\[
\rightarrow \mathbb{Z}
\]

Now we can "simplify F by making use of properties of \(\mu\)."
Since \( \bar{r} \) is going to zero all elements of \( F \) of the form

\[
\bar{r}(uv + au'v', w) = \bar{r}(uv) = a \bar{r}(v', w) \\
\bar{r}(v, w + aw') = \bar{r}(uv) = a \bar{r}(v, w')
\]

we might as well quotient \( F \) with the subspace generated by such elements: "\( A \)."

![Diagram](image)

\[ V \times W \xrightarrow{\alpha} F \xrightarrow{\beta} F/A \]

\[ \beta: f \mapsto f + A \text{ epic} \]

\[ A \subseteq \ker \bar{r} \]

\[ \bar{r} \] is unique because \( \beta \) is epi: (why?)

\[ F/A \cong V \otimes W \]

This is called the tensor product of \( V \) and \( W \).

More next time...

- More tensors
- Exterior algebra, determinants
- Eigenvalue problems, operator algebra
- Inner product spaces
- Applications