Lecture 1.

A category is a collection of “objects” such that for each pair \( A, B \) of objects we have a set of \( \text{morphisms} \) \( \text{Mor}(A, B) \). Morphisms can be composed

\[
\begin{align*}
A & \xrightarrow{\gamma} B & \quad & \xrightarrow{\psi} \quad & C \\
\quad & \downarrow \psi \circ \gamma & & \downarrow \quad & \\
\psi \circ \gamma & & & & 
\end{align*}
\]

The composition must be associative. Also, for each object \( A \), there must be an identity morphism \( \text{id}_A : A \rightarrow A \) such that \( \psi \circ \text{id}_A = \psi \) and \( \text{id}_B \circ \psi = \psi \) for any \( \psi, \psi' \).

**Examples**

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<td>Pointed sets</td>
<td>( (X, x) \xrightarrow{\gamma} (Y, y) ) s.t. ( \gamma(x) = y )</td>
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<tr>
<td>“Principle G-bundles”</td>
<td>( E \xrightarrow{\gamma} F ), ( \pi_E ) \text{ over } M, \pi_F</td>
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- \( \text{Y ex.} \)
  - Power set of a set
  - Dual of a vector space
  - Lie algebra of a Lie group
  - Homotopy \( \cong \neq \cong \)
  - Cohomology
  - “Second quantization”
  - ...are all the same categorical construction.

Categories

- \( \text{“Natural transformations”} \)
Distinguished Morphisms

A morphism \( A \xrightarrow{g} B \) is...

... a monomorphism if

\[
\begin{array}{c}
X \xrightarrow{k} A \xrightarrow{g} B \Rightarrow \alpha = \alpha' \\
\end{array}
\]

... an epimorphism if

\[
\begin{array}{c}
A \xrightarrow{g} B \ni X \Rightarrow \alpha = \alpha' \iff \\
\end{array}
\]

... an isomorphism if

\[
\begin{array}{c}
i_A \circ A \xrightarrow{g} B \ni \circ_g i_B \\
\end{array}
\]

commutes for some \( g^{-1} \). "\( A \cong B \)"

Category of sets

\( g \) is one-to-one
\( g \) is "injective"

\( g \) is onto
\( g \) is "surjective"

\( g \) has a two-sided inverse
\( g \) is "bijective"

Thenem. A function in set \( A \xrightarrow{g} B \)
is a monomorphism iff it is 1-1.

Proof. Suppose that \( g \) is monic.
Then

\[
\begin{array}{c}
\{13\} \circ A \xrightarrow{g} B \Rightarrow \alpha = \alpha' \\
\end{array}
\]

iff \( g(\alpha) = g(\alpha') \Rightarrow \alpha = \alpha' \Rightarrow g \) is 1-1.

Conversely, suppose that \( g \) is 1-1
and

\[
\begin{array}{c}
X \xrightarrow{k} A \xrightarrow{g} B \text{ commutes for some } \alpha \neq \alpha'. \\
\end{array}
\]

\( g(x) \neq g'(x') \) for some \( x \in X \Rightarrow y(g(x)) \neq y(g'(x)) \) since \( g \) is 1-1
\( \Rightarrow \) the diagram does not commute \( \Rightarrow \).
Simple Facts

Fact. If \( A \xrightarrow{\alpha} B \) and \( B \xrightarrow{\psi} C \) are monic, so is \( \psi \circ \alpha \).

Proof: Consider

\[
\begin{array}{c}
\times \xrightarrow{\alpha} A \xrightarrow{\psi} B \xrightarrow{\psi} C.
\end{array}
\]

If this commutes, then \( \psi \circ \alpha = \psi \circ \alpha' \) (since \( \psi \) is monic)

\( \Rightarrow \alpha = \alpha' \) (since \( \psi \) is also monic) \( \Rightarrow \psi \circ \alpha \) is monic.

+ miscellaneous similar results.

Products and Sums

Can we "treat a bunch of objects effectively as one object"?

\[
\begin{array}{c}
A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D
\end{array}
\]

\[
\begin{array}{c}
A \xleftarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D
\end{array}
\]

\[
\begin{array}{c}
\{ \text{fixed} \}
\end{array}
\]

A product "A \times B" of objects A, B consists of \( A \times B \) and two morphisms \( \alpha, \beta \)

\[
\begin{array}{c}
A \xleftarrow{\alpha} A \times B \xrightarrow{\beta} B
\end{array}
\]

\[
\begin{array}{c}
\times \xrightarrow{\times} A \times B \xrightarrow{\alpha} A \xrightarrow{\beta} B
\end{array}
\]

\[
\begin{array}{c}
\times \xrightarrow{\gamma} A \times B \xrightarrow{\psi} C
\end{array}
\]

\[
\begin{array}{c}
\times \xrightarrow{\gamma} A \times B \xrightarrow{\psi} C
\end{array}
\]

For any \( x, y, \psi \), there is a unique \( y \) s.t. the diagram commutes.

In the category of sets, \( A \times B \) is the cartesian product

\[
\begin{array}{c}
y(x) \xrightarrow{\gamma} (y(x), \psi(x)) \xrightarrow{\psi(x)} \psi(x)
\end{array}
\]

\[
\begin{array}{c}
y \xrightarrow{\gamma} x \xrightarrow{\psi(x)} \psi(x)
\end{array}
\]

\[
\begin{array}{c}
\alpha(a, b) = a \quad \beta(a, b) = b
\end{array}
\]

the diagram commutes for exactly one function

\[
\begin{array}{c}
\gamma : x \mapsto (y(x), \psi(x))
\end{array}
\]
Theorem: Products are unique in any category

Proof. Suppose \((A \times B, \alpha, \beta)\) and \((A \times B)', \alpha', \beta'\) are both products of \(A\) and \(B\).

\[
\begin{array}{ccc}
A & \xleftarrow{\alpha} & A \times B \\
\downarrow \alpha_A & & \downarrow \gamma \\
A & \xleftarrow{\alpha'} & (A \times B)' \\
\downarrow \alpha_A & & \downarrow \gamma' \\
A & \xleftarrow{\alpha} & A \times B
\end{array}
\]

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & B \\
\downarrow \beta_B & & \downarrow \gamma \\
B & \xrightarrow{\beta'} & B \\
\downarrow \beta_B & & \downarrow \gamma' \\
B
\end{array}
\]

\[
\begin{array}{ccc}
\text{commutes for a unique } \gamma' \\
\text{commutes for a unique } \gamma
\end{array}
\]

\[\Rightarrow\text{ the whole diagram commutes}\]

\[
\begin{array}{ccc}
A & \xleftarrow{\alpha} & A \times B \\
\downarrow \alpha_A & & \downarrow \gamma \circ \gamma' \\
A & \xleftarrow{\alpha} & A \times B
\end{array}
\]

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & B
\end{array}
\]

\[
\text{commutes}
\]

However this clearly also commutes with \(\alpha_{A \times B}\).

Since the product \(A \times B\) guarantees a unique morphism, \(\alpha_{A \times B} = \gamma \circ \gamma'\). Similarly \(\alpha'_{(A \times B)} = \gamma' \circ \gamma'\) \(\Rightarrow A \times B \cong (A \times B)'\).
Misc. definitions in set

A function \( f : A \rightarrow B \) is often specified informally by a rule such as \( f : a \mapsto a^2 \). Strictly speaking, a function is a subset of \( A \times B \)

\[
\{ (a, f(a)) : a \in A \}
\]

A relation \( R \) on a set \( A \) is just a subset of \( A \times A \).

\[ a R b \Leftrightarrow (a, b) \in R \]

Ex. The partial ordering \( \leq \) is a relation with:
\[ a \leq b \text{ and } b \leq c \Rightarrow a \leq c \] (transitive)
\[ a \leq b \text{ and } b \leq a \Rightarrow a = b \] (symmetric)
\[ a \leq a \] (reflexive)

For example, subsets of a set \( X \) are "partially ordered by inclusion" if \( A \subseteq B \iff A \subseteq B \cup A \subseteq X \).

An equivalence relation \( E \) has these properties:
\[ a E b \text{ and } b E c \Rightarrow a E c \]
\[ a E b \Rightarrow b E a \]
\[ a E a \]

Ex. \( = \) is an equivalence relation.

Ex. \( n E r' \Leftrightarrow n - r' = n \cdot 2\pi \) is an equivalence relation on \( \mathbb{R} \).
The equivalence class containing \( a \in A \) is defined to be

\[
[a] \equiv \{ x \in A : x \mathcal{E} a \}
\]

**Theorem.** Each \( a \in A \) is in exactly one equivalence class.

**Proof.** Let \( a \in A \). Then \( a \in [a] \) because \( a \mathcal{E} a \). If \( a \in [b] \) also, then \( a \mathcal{E} b \). \( \Rightarrow \) for any \( x \in [a] \), \( x \mathcal{E} a \), \( x \mathcal{E} b \Rightarrow x \in [b] \). Similarly \( [b] \subseteq [a] \Rightarrow [a] = [b] \).

The set of equivalence classes \( \mathcal{E} \) covers \( A \) without overlapping.

**Theorem.** If \( \mathcal{E}_1 \) are equivalence relations on \( A \), then \( \bigwedge \mathcal{E}_1 \) is also an equivalence relation on \( A \).

**Proof.** Easy.

**Def.** The equivalence relation generated by relation \( R \in \mathcal{A} \times \mathcal{A} \)

\[
\text{Eq}(R) = \{ \text{intersection of all equivalence relations that contain } R \}
\]

**Example:**

A relation \( R \in \mathcal{A} \times \mathcal{A} \) is just the (solid) edges of some graph on \( A \).

\( \text{Eq}(R) \) is the relation

\[ a \mathcal{E}(R) b \text{ iff there is a path from } a \text{ to } b. \]

\( A/\text{Eq}(R) \) are just the connected components.
Even in set, there are non-trivial applications

Jet finding in HEP

Clustering in a pixel detector

Finding contiguous spins on a lattice

\[ A \] = Momentum vectors
\[ A \] = Cells
\[ A \] = Lattice sites

\[ p R q \] if \( q \) is the "nearest" to \( p \)

\[ c R d \] if cell \( c \) is the "nearest" to \( d \).

\[ i R j \] if \( i \) and \( j \) are adjacent and \( \text{spin}(i) = \text{spin}(j) \).

\[ \frac{A}{E_q(R)} \equiv \text{jets} \]
\[ \frac{A}{E_q(R)} \equiv \text{clusters} \]
\[ \frac{A}{E_q(R)} \equiv \text{contiguous spin groups} \]

All are solved by the same \( N \)-logn (\( N \equiv |A| \)) algorithm: "Finding the connected components of a digraph."