Mathematical Physics Homework 4. Topology

1. Considering complements with respect to some fixed set \( X \), prove that the complement of an arbitrary union of sets is the intersection of the complements of those sets.

2. Consider \( A \subseteq B \). Does \( A \subseteq A' \) imply \( \varphi[A] \subseteq \varphi[A'] \)? Does \( B \subseteq B' \) imply \( \varphi^{-1}[B] \subseteq \varphi^{-1}[B'] \)?

3. Prove that closed sets have the properties claimed on page 1 of the notes.

4. Prove that if objects are topological spaces and morphisms are continuous mappings, that this is a category.

5. Prove that the open sets of a metric space as defined in example 4 on page 2 is, in fact, a topology.

6. Prove that every open subset of the real line is a union of open intervals.

7. Prove that every subset of the real line is an intersection of open sets (it’s easier than it sounds).

8. Let \( \mathbb{R} \xrightarrow{\varphi} \mathbb{R}^2 \) be given by \( \varphi(r) = (\cos r, \sin r) \). Show that \( \varphi \) is continuous.

9. Find an isomorphism from the subspace \((0,1)\) of the real line to \( \mathbb{R} \).

10. Work out the details of the decomposition of \( X \xrightarrow{\varphi} Y \) on page 7 of the notes.

11. Prove that monics in the category of topological spaces are 1-1 functions and epics are onto functions. Prove that the characterization of isomorphisms on page 7 of the notes is correct.

12. If \( A \) is a subset of topological space \( X \), show that in the standard inherited topology, closed sets in \( A \) are sets of the form \( A \cap C \) with \( C \) closed in \( X \).

13. Suppose that \( A \) is a collection of subsets of a set \( X \) which is closed under pairwise intersection and which includes both \( X \) and the empty set. Prove that arbitrary unions of sets in \( A \) is the topology generated by \( A \).

14. The direct product of topological spaces \( X \) and \( Y \) is a certain topology on the cartesian product \( X \times Y \). This space is defined as the topology generated by sets \( \alpha^{-1}[O_X] \) and \( \beta^{-1}[O_Y] \) where \( \alpha \) and \( \beta \) are the standard projections and \( O_X \) and \( O_Y \) are open sets in \( X \) and \( Y \) respectively. Geroch then notes that this topology is arbitrary unions of sets \( \alpha^{-1}[O_X] \cap \beta^{-1}[O_Y] \). Why is this true?

15. For navigation purposes, it would be convenient to have a continuous “smooth” invertible mapping from the surface of the earth to the plane. Is this possible? If not, why not?
16. Let $X$ be a set with the discrete topology. When is $X$ compact?

17. Prove that $\mathbb{R}$ does not have the same topology as $\mathbb{R}^2$ (hint: consider “removing a point from $\mathbb{R}$”).

18. Prove that the real line is connected (see Geroch if you get stuck).