Vector Spaces (S. Younus) Chapter Ch. 9, 10, 11, 13

A real (complex) vector space \( V \) is an abelian group with bilinear product \( \mathbb{R} \times V \to V \) \( (c \times v + a \times v') \to V \) satisfying \( (a + c)v = a(v) + c(v) \) and \( 1 \times v = v \).

The category of real (complex) vector spaces consists of

Objects: real (complex) vector spaces

Morphisms: linear maps \( V \to W \) \( \varphi(v) = \varphi(v) + \varphi(v) \)

In this category

\( \text{monomorphisms } \iff \text{one-to-one linear maps } \iff \text{Ker } \varphi = \{ 0 \} \)

\( \text{epimorphisms } \iff \text{onto linear maps } \iff \text{Im } \varphi = W \)

\( \text{isomorphisms } \iff \text{bi-invertible linear maps } \iff \text{Ker } \varphi = \{ 0 \} \) and \( \text{Im } \varphi = W \).

The cartesian product \( V \times W \) with the obvious vector space structure is denoted \( V \oplus W \) and is both direct sum and direct product. 

\( \text{Mor } (V, W) \) is also a vector space with \( (f + g)(v) = f(v) + g(v) \) etc.

A free construction

\[ S \xleftarrow{\alpha} V \xrightarrow{\beta} W \]

\( \varphi(f) = \sum_{s \in S} \varphi(s) f(s) \)

Subset \( K \) of \( V \) is independent if \( a_j k_j = 0 \Rightarrow a_j = 0 \) \([\text{finite sum over repeated indices}].\)

Subset \( K \) of \( V \) spans \( V \) if \( v = a_j k_j \) for all \( v \in V \).

Subset \( K \) of \( V \) is a basis if \( K \) is independent and spans \( V \).

A vector space \( V \) has a basis \( \iff V \) is the free vector space on some set.

Theorem: Every vector space has a basis.

Theorem: For free vector spaces \( S \to V \) and \( S' \to V' \), \( \text{uses Zorn's lemma } \) \( V \cong V' \Rightarrow S \cong S' \).
Subspaces

Just as with groups, \( W \subseteq V \) is called a subspace if it is a vector space in its own. Also, just as with groups, if \( V \subseteq U \), ker \( y \) and \( \text{Im} \ y \) are subspaces. All subspaces are normal, so \( V/W \) always exists.

If \( W = \ker y \), then \( y \) is also a monomorphism. If \( y \) is also epi, then \( V/W \cong Z \).

Example: \( V = \mathbb{R}^3 \)

\( W \) is a subspace

\( V/W \) is the set of lines parallel to \( W \)

\( V/W \cong x-y \) plane

Subspaces \( U \) and \( W \) are complementary in \( V \) if, for every \( v \in V \), \( v = u + w \) for some \( u \in U \), \( w \in W \), and if \( u + w = 0 \) \( \Rightarrow u = w = 0 \).

Theorem (Gersch): Complementary subspaces exist.

Theorem: If \( U \) and \( W \) are complementary in \( V \), then \( V \cong U \oplus W \).

Proof. Let linear \( y : (u,w) \mapsto u+w \) be a mapping from \( U \oplus W \) to \( V \).

Since \( U \) and \( W \) are complementary, \( y \) is epi. Suppose \( y((u,w)) = y((u',w')) \). Then \( u+w = u'+w' \Rightarrow (u-u') + (w-w') = 0 \Rightarrow u = u', w = w' \Rightarrow y \) is mono \( \Rightarrow V \cong U \oplus W \).

Example: \( V = \text{Mat} \left( \mathbb{C}^{2 \times 2}, \mathbb{C} \right) \), \( A \subset \text{Mat} \left( \mathbb{C}^{2 \times 2}, \mathbb{C} \right) \)

\( U = \{ f \in V : f(x) = 0 \text{ for } x \in A \} \) \( \text{U} \) and \( W \) are complementary

\( W = \{ f \in V : f(x) = 0 \text{ for } x \notin A \} \)
Georg No. 83.

Given $\alpha, \beta$, is there a morphism $\lambda$ as indicated that causes the triangle to commute?

If so, when is it unique?

**Answer:** Such a morphism exists if and only if $\ker \alpha \subseteq \ker \beta$.

**Proof.** Suppose $V \xrightarrow{\gamma} W$ causes the diagram to commute.

Then $\ker \beta = \ker(\alpha \circ \gamma) \subseteq \ker \alpha$.

Suppose that $\ker \alpha \subseteq \ker \beta$. Define a morphism

$\lambda : \mathrm{im} \alpha \to \mathrm{im} \beta$ by

$\lambda : \alpha(u) \mapsto \beta(u)

Notice that $\alpha(u) = \alpha(u') \Rightarrow u - u' \in \ker \alpha \Rightarrow u - u' \in \ker \beta

\Rightarrow \beta(u) = \beta(u') \Rightarrow \lambda(\alpha(u)) = \lambda(\alpha(u')) \Rightarrow \lambda$ is a function.

$\lambda$ is also linear since $\lambda(\alpha(u) + \alpha(u')) = \lambda(\alpha(u + u')) = \beta(u) + \beta(u') = \lambda(\alpha(u)) + \lambda(\alpha(u'))$.

Let $V' \cong \text{im} \alpha \oplus V'$ where $V'$ is complementary to $\text{im} \alpha$.

$W \cong \text{im} \beta \oplus W'$ where $W'$ is complementary to $\text{im} \beta$.

Consider

$\text{im} \alpha \rightarrow \text{im} \alpha \oplus V' \leftarrow V'$

$x \downarrow \quad \downarrow x$.

$\text{im} \beta \rightarrow \text{im} \beta \oplus W' \leftarrow W'$

Since there is always a morphism $x : V' \rightarrow W'$, $x$ solves the problem. Also, there will be more than one $x$ unless there is only one $x : V' \rightarrow W'$, e.g. if $V' = \{a\} \times W, W \times \{b\}$, e.g. when $\alpha \circ \beta$ is epi.

If $\beta$ is epi, $x$ is unique. Also, if $\alpha$ is epi, $\lambda$ is uniquely determined by the triangle commuting. QED.
Duals

Given a vector space \( V \), \( \text{M} \nu (V, \mathbb{R}) \) \("V^*"\) is called the dual of \( V \). Notice that because of the free construction,

\[
\begin{array}{ccc}
S & \rightarrow & V \\
\downarrow f & & \downarrow g \\
\rightarrow & & \rightarrow \\
& & \mathbb{R}
\end{array}
\]

\( f \mapsto g \) is an isomorphism \( \text{M} \nu (S, \mathbb{R}) \cong V^* \).

Given \( V \rightarrow W \), the dual lets us also define \( V^* \leftarrow W^* \) by

\[ g^*: f \mapsto f \circ g \]

Example: Let \( V \) be the vector space of continuous functions from \( C_0, [0,1] \) to \( \mathbb{R} \). For each \( m \in V \), we can define

\[ f \mapsto \int_0^1 f(x) m(x) \, dx \quad V \rightarrow \mathbb{R} \text{ i.e. } \in V^* \]

\( V \) is isomorphic to such elements, but this is not all of \( V^* \! \)

For example, for any fixed \( a \in C_0, [0,1] \),

\[ f \mapsto f(a) \]

is linear, but this element of \( V^* \) is not \( \int_0^1 f(x) m(x) \, dx \) for any \( m \in C_0, [0,1] \) [it would for \( m(x) = \delta(x-a) \), but \( \delta(x-a) \) is not a function].

For finite dimensional vector spaces, \( V \cong V^{**} \). In general, Grothendieck proves that \( V \cong V^{**} \) iff \( V \) is finite dimensional.
Exercise #83.

Show that \((V \oplus W)^* \cong V^* \oplus W^*\).

Consider the direct sum

\[
\begin{array}{c}
V \xrightarrow{\alpha} V \oplus W \xleftarrow{\beta} W \\
\downarrow \quad \downarrow \gamma \quad \downarrow \delta \\
\quad f \quad \quad \quad \quad \quad R \quad \quad \quad \quad \quad g
\end{array}
\]

This constitutes an invertible map \(\Phi : (f, g) \mapsto \alpha f + \beta g\) from \(V^* \oplus W^*\) to \((V \oplus W)^*\). \(\Phi\) is also linear, so \(V^* \oplus W^* \cong (V \oplus W)^*\).
Example.

Suppose that \( V \xrightarrow{\phi} W \) is a monomorphism. We want to show that \( \gamma^*: W^* \rightarrow V^* \), \( \gamma^*: f \mapsto f \circ \phi \) is epi. Let \( \gamma \) be any element of \( V^* \).

Since \( \gamma \) is mono, \( \ker \gamma \subseteq \ker \phi \), a \( \bar{\phi} \) exists s.t.

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\downarrow \phi & & \downarrow \bar{\phi} \\
\gamma & \rightarrow & \gamma \\
\end{array}
\]

commutes. Then \( \gamma = \gamma^*(\bar{\phi}) \Rightarrow \gamma^* \) is epi.

On the other hand, if \( \gamma \) is epi, then

\[
\begin{array}{ccc}
V & \xrightarrow{\phi} & W \\
\downarrow \phi & & \downarrow \phi \\
\gamma^*(\phi) & \rightarrow & \gamma^* \\
\end{array}
\]

commutes, \( \Rightarrow \phi \Rightarrow \gamma^* \) is monomorphic.
Tensor Products (Define: Geometric).

Given multilinear $V \times W \overset{\alpha}{\to} Z$, let $V \times W \overset{\beta}{\to} F$ be the free vector space on $V \times W$ and let $A$ be the subspace generated by

$$
\begin{align*}
\alpha(v + av', w) &= \alpha(v, w) - \alpha(v', w) \\
\alpha(v, w + aw') &= -\alpha(v, w) - \alpha(v, w')
\end{align*}
$$

for $v, v' \in V$, $w, w' \in W$, $a \in \mathbb{R}$. Consider

$$
V \times VV \overset{\alpha}{\to} F
$$

Since $\alpha$ is linear and $\alpha(a) = 0$ for any $a \in A$, $\alpha$ is also zero on any $a' \in A$.

Thus

$$
\begin{array}{ccc}
V \times W & \overset{\alpha}{\to} & F \\
\downarrow & & \downarrow \phi \phi' \\
\downarrow & & \downarrow \\
V \times W & \overset{\beta}{\to} & F/A
\end{array}
$$

A $\phi \circ \phi'$ is a $\phi'$ s.t. the right triangle commutes. Since $\phi$ is of $\phi'$, $\phi'$ is unique. $\Rightarrow$ The whole diagram commutes.

It only remains to show that $\beta \circ \phi$ is bilinear.

$$
\begin{align*}
\beta \circ \phi\left(\alpha(v + av', w)\right) &= \beta(\alpha(v, w) - \alpha(v', w)) \\
\beta \circ \phi\left(\alpha(v, w + aw')\right) &= -\beta(\alpha(v, w) - \alpha(v, w'))
\end{align*}
$$

Thus $\beta \circ \phi$, $F/A$ is the tensor product of $V \times W$. This is conventionally renamed

$$
V \times W \overset{\otimes}{\to} V \otimes W
$$