Consensus and Deadlock in Opinion Dynamics

Sidney Redner, CNLS, LANL (on leave from Boston University)
collaborators: E. Ben-Naim, P. Chen, P. Krapivsky, V. Sood, F. Vazquez

Basic questions: What is the final state in prototypical opinion dynamics models with primarily ferromagnetic interactions?

How long does it take the reach the final state?

Models: Voter model on heterogeneous graphs

Majority rule

Bounded compromise models

Spiteful extremists & accommodating centrists

Basic results: Voter model: fast consensus on heterogeneous graphs

Majority rule: multiscale dynamics & slow consensus

Bounded compromise: rich political bifurcation sequence

Spiteful extremists: consensus versus deadlock
Voter Model

0. Binary spin variable at each site
1. Pick a random spin
2. Assume state of randomly-selected neighbor
   *each individual has zero self-confidence and adopts state of randomly-chosen neighbor*
3. Repeat 1 & 2 until consensus *necessarily* occurs

Example update step:
Voter model on regular lattices

1. Final state (exit) probabilities

follows from magnetization conservation

2. Dependence of consensus time on system size:

<table>
<thead>
<tr>
<th>dimension</th>
<th>consensus time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$N^2$</td>
</tr>
<tr>
<td>2</td>
<td>$N \ln N$</td>
</tr>
<tr>
<td>$&gt;2$</td>
<td>$N$</td>
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Voter model on heterogeneous graphs

Illustrative example: complete bipartite graph

\[ dN_a = \frac{a}{a+b} \left[ \frac{a - N_a}{a} \frac{N_b}{b} - \frac{N_a}{a} \frac{b - N_b}{b} \right] \]

\[ dN_b = \frac{b}{a+b} \left[ \frac{b - N_b}{b} \frac{N_a}{a} - \frac{N_b}{b} \frac{a - N_a}{a} \right] \]

Subgraph densities: \( \rho_a = \frac{N_a}{a}, \rho_b = \frac{N_b}{b} \quad dt = 1/(a+b) \)

\[ \rho_{a,b}(t) = \frac{1}{2} [\rho_{a,b}(0) - \rho_{b,a}(0)] e^{-2t} + \frac{1}{2} [\rho_a(0) + \rho_b(0)] \]

\[ \rightarrow \frac{1}{2} [\rho_a(0) + \rho_b(0)] \quad \text{N.B.: magnetization is not conserved} \]

Exit probabilities:

\[ E_+ = 1 - E_- = \frac{1}{2} [\rho_a(0) + \rho_b(0)]. \]

Exit probabilities

\[ E_+ = 1 - E_- = \frac{1}{2} \left[ \rho_a(0) + \rho_b(0) \right] \]

extreme case: star graph

initial state: 1 plus, N minus

final state: all + with probability 1/2!
Mean consensus time

\[ T(\rho_a, \rho_b) = \frac{a}{a + b} (1 - \rho_a) \rho_b [T(\rho_a + \frac{1}{a}, \rho_b) + \delta t] \]
\[ + \frac{a}{a + b} \rho_a (1 - \rho_b) [T(\rho_a - \frac{1}{a}, \rho_b) + \delta t] \]
\[ + \frac{b}{a + b} (1 - \rho_b) \rho_a [T(\rho_a, \rho_b + \frac{1}{b}) + \delta t] \]
\[ + \frac{b}{a + b} \rho_b (1 - \rho_a) [T(\rho_a, \rho_b - \frac{1}{b}) + \delta t] \]
\[ + (1 - \rho_a - \rho_b + 2 \rho_a \rho_b) [T(\rho_a, \rho_b) + \delta t], \]

continuum limit:

\[ N \delta t = (\rho_a - \rho_b) (\partial_a - \partial_b) T(\rho_a, \rho_b) \]
\[ - \frac{1}{2} (\rho_a + \rho_b - 2 \rho_a \rho_b) \left( \frac{1}{a} \partial_a^2 + \frac{1}{b} \partial_b^2 \right) T(\rho_a, \rho_b) \]
Trajectories of single voter model realizations

$t < 1$

complete bipartite graph

two-clique graph

$K_{10000}$

$c = 100$

$c = 1$
assuming $\rho_a = \rho_b$ and $\rho = (\rho_a + \rho_b)/2$:

equation of motion for $T$ becomes:

$$N\delta t = (\rho_a - \rho_b)(\partial_a - \partial_b)T(\rho_a, \rho_b) - \frac{1}{2}(\rho_a + \rho_b - 2\rho_a\rho_b)\left(\frac{1}{a}\partial_a^2 + \frac{1}{b}\partial_b^2\right)T(\rho_a, \rho_b)$$

$$\frac{1}{4}\rho(1 - \rho) \left(\frac{1}{a} + \frac{1}{b}\right) \partial^2 T = -1$$

with solution:

$$T_{ab}(\rho) = -\frac{4ab}{a + b} \left[(1 - \rho) \ln(1 - \rho) + \rho \ln \rho\right]$$

implication: $a = \mathcal{O}(1)$, $b = \mathcal{O}(N)$ (star graph), $T = \mathcal{O}(1)$

$a = \mathcal{O}(N)$, $b = \mathcal{O}(N)$ (symmetric graph), $T = \mathcal{O}(N)$
Arbitrary degree distribution network

\[ n_j = \text{fraction of nodes with degree } j \]

\[ \mu_m = \sum_j j^m n_j = m^{\text{th}} \text{ moment of degree distribution} \]

\[ \omega = \frac{1}{\mu_1} \sum_j jn_j \rho_j = \text{degree-weighted up spin density} \]

Basic result:
\[ T_N(\omega) = -N \frac{\mu_1^2}{\mu_2} \left[ (1 - \omega) \ln(1 - \omega) + \omega \ln \omega \right] \]

For power-law network:
\[ n_j \sim j^{-\nu} \]

\[ T_N \sim \begin{cases} 
N & \nu > 3, \\
N / \ln N & \nu = 3, \\
N^{(2\nu-4)/(\nu-1)} & 2 < \nu < 3, \\
(\ln N)^2 & \nu = 2, \\
\mathcal{O}(1) & \nu < 2.
\end{cases} \]
Consensus times for power-law degree distributions \( n_j \sim j^{-\nu} \)

Basic results: quick consensus! universal scaling
Majority rule

1. Pick a random group of $G$ spins (with $G$ odd).
2. All spins in $G$ adopt the majority state.
3. Repeat until consensus necessarily occurs.

```
+ + - -  
- + + -  
+ - - +  
+ - + +  

+ + + -  
+ + + -  
+ + + +  
+ - + +  
```

Basic questions:  
1. Which final state is reached? 
2. What is the time until consensus?
Mean-field theory (for G=3)

\[ E_n \equiv \text{exit probability to } m = 1 \text{ starting from } n \text{ plus spins} \]
\[ = p_n E_{n+1} + q_n E_{n-1} + r_n E_n \]

where
\[ p_n = \frac{\binom{3}{2} (N-3)}{n-2} \binom{n}{n} \]
\[ q_n = \frac{\binom{3}{1} (N-3)}{n-1} \binom{n}{n} \]
\[ r_n = 1 - p_n - q_n \]

\[ T_n \equiv \text{mean time to } m = 1 \text{ starting from } n \text{ plus spins} \]
\[ = p_n (T_{n+1} + \delta t) + q_n (T_{n-1} + \delta t) + r_n (T_n + \delta t) \]
Exit probability
(schematic)

Consensus time
(data)

1
E(m)
0
-1

N

-1/2

1
0
1
m

1.0
0.0
1.0

0.0
0.5
1.0

T_m/(2 lnN)

---N=81
--- 401
--- 2001
--- 10001
--- 50001
Consensus time for finite spatial dimensions

Critical dimension appears to be $>4$!
Anomalous dynamics in 2d: stripes \(~33\%\) of the time!
Slab formation in 3d ~8% of the time
Consensus time distribution

multiscale relaxation
to final consensus

2d

3d

4d

multiscale relaxation
to final consensus
Bounded compromise model

\[ x_1 \rightarrow x_2 \rightarrow x_1 \]

If \(|x_2 - x_1| < 1\) compromise

\[ \frac{x_1 + x_2}{2} \]

If \(|x_2 - x_1| > 1\) no interaction
Master equation

Fundamental parameter: $\Delta$, the initial opinion range

Basic observable: $P(x,t) = \text{probability that agent has opinion } x$

\[
\frac{\partial P(x, t)}{\partial t} = \int \int_{|x_1 - x_2| < 1} dx_1 dx_2 \ P(x_1, t)P(x_2, t) \times \left[ \delta \left( x - \frac{x_1 + x_2}{2} \right) - \delta(x - x_1) \right]
\]

$\Delta < 1$: eventual consensus

$\Delta > 1$: disjoint “parties”
Early time evolution (for $\Delta=4$)
Early time evolution (for $\Delta=10$)
Cluster masses versus $\Delta$

- Central
- Major
- Minor

Equations:
$$(\Delta - \Delta_c)^4$$
$$(\Delta - \Delta_c)^3$$
Minor cluster bifurcations

major cluster: $w \approx e^{-t/2}$

minor cluster: $\dot{m} = -m$

$\rightarrow m(t) = m(0) e^{-t} = \epsilon e^{-t}$

separation:

$w = \epsilon = e^{-t_{\text{sep}}/2}$

$\rightarrow m(t_{\text{sep}}) \propto \epsilon^3$
Cluster masses near bifurcations
Spiteful extremist model


0. 3-state variable at each site: $- 0 +$
1. Pick a random spin
2. Assume state of neighbor if compatible
3. Repeat until either consensus or frozen final state

\[+0 \quad \leftrightarrow \quad ++ \quad \leftrightarrow \quad 00\]
\[-0 \quad \leftrightarrow \quad -- \quad \leftrightarrow \quad 00\]

compatible

\[+- \quad \rightarrow \quad ++ \quad \text{incompatible}\]
Evolution in composition space

\[ p_x, y = \frac{N \pm N_0}{N(N-1)} \]

consensus states

frozen states
Probability to reach frozen final state

\[ F(x, y) = \text{probability to reach frozen state from } (x, y) \]

recursion formula:

\[
F(x, y) = p_x[F(x - \delta, y) + F(x + \delta, y)] \\
+ p_y[F(x, y - \delta) + F(x, y + \delta)] \\
+ [1 - 2(p_x + p_y)]F(x, y)
\]

continuum limit:

\[
x \frac{\partial^2 F(x, y)}{\partial x^2} + y \frac{\partial^2 F(x, y)}{\partial y^2} = 0, \quad F(x, 0) = 0, \quad F(0, y) = 0, \quad F(x, 1 - x) = 1
\]

solution:

\[
F(x, y) = \sum_{n \text{ odd}} \frac{2(2n + 1)}{n(n + 1)} \sqrt{xy} (x + y)^n P_n^1 \left( \frac{x - y}{x + y} \right)
\]
Final state probabilities

Final state probabilities = density of 0
In each region, the probability of reaching specified final state is >50%

moral: extremism promotes deadlock
Outlook & some open questions

1. Heterogeneous voter model: fast consensus
   - What is the route to consensus?
   - Role of fluctuations?
   - Behavior of the correlations?

2. Majority rule: complex relaxation to a simple final state
   - Why do stripes occur?
   - What is the critical dimension?
   - What happens for more than 2 states?

3. Bounded compromise: rich bifurcation sequence

4. Spiteful extremists: deadlock & multiple final states
   - Implications for real politics?