1. We consider transverse vibrations of a planar square lattice of rows and columns of identical atoms, and let $u_{l,m}$ denote the displacement normal to the plane of the lattice of the atom in the $l$th column and $m$th row (Fig. 1). The mass of each atom is $M$, and $C$ is the force constant for nearest neighbor atoms.

(a) Show that the equation of motion is (5 points)

$$M(d^2u_{l,m}/dt^2) = C\left((u_{l+1,m} + u_{l-1,m} - 2u_{l,m}) + (u_{l+1,m+1} + u_{l,m-1} - 2u_{l,m})\right).$$

(b) Assume solutions of the form

$$u_{l,m} = u(0) \exp[i(lK_xa + mK_ya - \omega t)]$$

where $a$ is the spacing between nearest-neighbor atoms. Show that the equation of motion is satisfied if (10 points)

$$\omega^2 M = 2C (2 - \cos K_xa - \cos K_ya).$$

This is the dispersion relation for the problem.

(c) Show that the region of $K$ space for which independent solutions exist may be taken as a square of side $2\pi/a$. (2 points) This is the first Brillouin zone of the square lattice. Sketch $\omega$ versus $K$ for $K = K_x$ with $K_y = 0$ (3 points), and for $K_x = K_y$. (3 points)

(d) For $Ka \ll 1$, show that (5 points)

$$\omega = (Ca^2/M)^{1/2} (K_x^2 + K_y^2)^{1/2} = (Ca^2/M)^{1/2} K,$$

So that in this limit the velocity is a constant.

2. Consider a cubic crystal where atoms of mass $M_1$ lies on one set of planes and atoms of mass $M_2$ lie on planes interleaved between those of the first set (Fig. 2). Let $a$ denote the repeat distance of the lattice in the direction normal to the lattice planes considered. We consider longitudinal waves that propagate in a symmetry direction for which a single plane contains only a single type of ions as the one illustrated in Fig. 2. (a) Write down the equation(s) of motion (10 points) and hence solve for the dispersion relations, $\omega(k)$ and eigen states of the normal modes.
(15 points) (b) Find the amplitude ratios $u/v$ ($u =$ amplitude of vibration modes of $M_1$, and similarly $v$ is for $M_2$) for the two branches at $K_{\text{max}} = \pi/a$. (6 points) Show that at this value of $K$ the two lattices act as if decoupled: one lattice remains at rest while the other lattice moves. (6 points)

![Fig. 2](image)

3. (a) Consider a dielectric crystal made up of layers of atoms, with rigid coupling between layers so that the motion of the atoms is restricted to the plane of the layer. Show that the phonon heat capacity in Debye approximation in the low temperature limit is proportional to $T^2$. (20 points)) (b) Suppose instead, as in many layer structures, that adjacent layers are very weakly bound to each other. What form would you expect the phonon heat capacity to approach at extremely low temperatures? (20 points)

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