Maxwell’s equations

James Clerk Maxwell wrote down a set of four equations that are a great summary of what we have done in this course so far. The equations also predict that electromagnetic (EM) waves with certain characteristics that we will discuss here can exist.

Maxwell’s equations:

Equation 1: electric fields come from charges.
Equation 2: magnetic field lines are continuous loops.
Equation 3: a changing magnetic flux can produce an electric field.
Equation 4: magnetic fields come from currents, or from changing electric flux (this was what Maxwell added).

Prediction of EM Waves

Following the Maxwell equations, an EM wave carries simultaneously oscillating electric E and magnetic B field, which can be written as:

\[ E(x,t) = E_0 \cos(kx - \omega t) \]
\[ B(x,t) = B_0 \cos(kx - \omega t) \]

This means that the E and B fields are in phase. The equations also predict that the EM wave travels at the speed, \( c \), given by:

\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.998 \times 10^8 \text{ m/s} \]

Soon it is realized that (optical) light is a kind of EM wave so \( c \) is the speed of light.

Another consequence of Maxwell’s equations is:

\[ \frac{c}{B} \]

EM waves do not need a medium. In some sense they keep themselves going - the oscillating electric field generates an oscillating magnetic field, and the oscillating magnetic field in turn generates an oscillating electric field.

To apply the right-hand-rule, point your fingers in the direction of the E vector then curl them towards the direction of the B vector. The direction your thumb pointing is the direction of travel of the wave.
Detecting EM waves

In an EM wave, the energy is divided equally between the electric fields and the magnetic fields. Usually, an antenna detects either the electric fields or the magnetic fields. Antennas are generally either conducting rods or conducting loops. Which would be used to detect the different fields?

1. Conducting rods are good for detecting electric fields and conducting loops are good for detecting magnetic fields
2. Conducting rods are good for detecting magnetic fields and conducting loops are good for detecting the electric fields
3. You can use either type of antenna for either type of field - you just need to orient the antenna appropriately.

Root-Mean-Square Average of the E and B Field

It is common to refer to the root-mean-square (r.m.s.) average of the electric and magnetic fields of an EM wave. They are the square root of the time average of \( E(t)^2 \) and \( B(t)^2 \). We are interested in the average of \( E(t)^2 \) and \( B(t)^2 \) rather than the average of \( E(t) \) and \( B(t) \) because their average value is always zero. (Why?)

\[
E_{\text{rms}} = \sqrt{\langle E(t)^2 \rangle} = \sqrt{\langle (E_0 \sin \omega t)^2 \rangle} = \sqrt{E_0^2/2} = E_0/\sqrt{2}
\]

\[
B_{\text{rms}} = \sqrt{\langle B(t)^2 \rangle} = \sqrt{\langle (B_0 \sin \omega t)^2 \rangle} = \sqrt{B_0^2/2} = B_0/\sqrt{2}
\]

In the above, \( \langle \ldots \rangle \) means average over time \( t \). Also note that \( \langle \sin^2(\omega t) \rangle = \frac{1}{2} \). \( E_{\text{rms}} \) and \( B_{\text{rms}} \) serves as the average value of the E and B field of the EM wave.

Average Energy Density carried by an EM Wave

\[
u = \frac{\text{Total energy}}{\text{Volume}} = \frac{1}{2} \epsilon_0 \langle E(t)^2 \rangle + \frac{1}{2} \mu_0 \langle B(t)^2 \rangle
\]

Recall \( \frac{E}{B} = c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \Rightarrow \epsilon_0 E^2 = \frac{B^2}{\mu_0} \)

Thus, \( \nu \approx \epsilon_0 \langle E(t)^2 \rangle = \frac{1}{2} \mu_0 \langle B(t)^2 \rangle = E_0^2/2 = B_0^2/2 \)

Intensity carried by an EM Wave in air or vacuum

Average Intensity (i.e., average power carried across a unit area):

\[
I = \frac{P}{A} = \frac{\text{Average total energy}}{t \times A} = \frac{\text{Energy Per volume}}{\text{Volume illuminated by the EM wave in time } t \text{ over an area } A}
\]

From the expressions of \( \nu \) in a previous page,

\[
I = c \nu, E_0^2/2 = c B_0^2/(2 \mu_0)
\]
Radiation Pressure of an EM Wave
Besides energy, EM waves carry momentum, \( p \), too.
For objects with a mass, 
\[ E = \frac{mv^2}{2} \quad \text{and} \quad p = mv \Rightarrow p = \frac{2E}{v} \]
For EM waves, which doesn’t have a mass, 
\[ p = \frac{E}{c} \quad \text{(a result of relativity)} \]
When an EM wave impinges on an object, it transfers momentum to the object. As a result, the object “feels” a pressure from the wave, which we refer to as the radiation pressure, \( P \).

If the object is 100% absorbing, 
\[ P = \frac{1}{c} \]
If the object is 100% reflecting, 
\[ P = \frac{2I}{c} \]
The factor 2 in the second case can be understood because if the EM wave is 100% bounced off the object, the change in momentum is twice as large than when there is no bouncing off at all.

Doppler effect of EM waves
Unlike sound waves, the speed of EM waves as seen by an observer is always the same irrespective of how he/she is moving. This has to do with relativity. As a result, there is no Doppler effect for EM waves associated with motions of the observer.

But if the source moves, the apparent wavelength would still be changed as in sound waves. This leads to the following equation for the Doppler effect of EM waves:
\[ f' = f \left(1 \pm \frac{v_{\text{rel}}}{c} \right) \]
Where \( v_{\text{rel}} \) is the relative velocity of the source to the observer. The + (-) sign applies when the source is approaching (receding from) the observer.

The electromagnetic spectrum
Visible light is a kind of electromagnetic wave. Below is a section of the electromagnetic spectrum encompassing the visible spectrum. Which side (left or right is the wavelength longer)?

\[ \text{green} \quad \text{blue} \quad \text{yellow} \quad \text{red} \]

Interactions with the human body
Gamma rays and X-rays are known as ionizing radiation. They can cause chemical changes as well as mutations of DNA.

Ultraviolet light is associated with suntans, sunburns, and cataracts.

Rods and cones in our eyes are sensitive to visible light, which is why we can see.

Heat sensors in our skin can detect infrared waves.

We’re not particularly sensitive to anything with a longer wavelength.