Gravitation

**Definition of Weight Revisited**

The weight of an object on or above the earth is the gravitational force that the earth exerts on the object. The weight always points toward the center of mass of the earth.

On or above another astronomical body, the weight is the gravitational force exerted on the object by that body. The direction of the weight (or gravitational force) points towards the center of mass of that body.

**SI Unit of Weight:** Newton (N)

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**Newton’s Law of Universal Gravitation**

\[ W = G \frac{M_E m}{r^2} \]  \hspace{1cm} (1)

where \( W \) is the weight of an object with mass \( m \) due to the earth’s gravitational force, \( G \) is the universal gravitational constant = \( 6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2 \), \( M_E \) is the mass of the earth, \( r \) is the distance between the object and the center of mass of the earth.

Write \( W \) in the usual form, \( W = mg \)

We get \( g = G \frac{M_E}{r^2} \)

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**Gravitational Field**

The gravitational field, \( g \), at a point is the gravitational force an object experiences when placed at that point divided by the object’s mass. For gravitational field coming from the earth, \( g = G \frac{M_E}{r^2} \frac{1}{m} \)

where \( g \) is in units of \( \text{m/s}^2 \) and \( r \) is the distance the point is from the center of mass of the earth. This result shows that the gravitational field is the same as the gravitational acceleration.

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**Gravitational Field**

Since the gravitational field is essential the gravitational force experienced by a unit mass at the point of interest, it should have a direction. The direction of the gravitational field is pointed towards the body that produces the field. In other words, gravitational force always attracts the object towards the body producing the field.

Note that gravitational force is a kind of interaction forces. So both the object and the body involved experience the same magnitude of attractive force from each other.
Gravitational field produced by \( m_2 \) at a distance of \( r_{12} \):

\[
F_{12} = m_1 g_2
\]

Gravitational field produced by \( m_1 \) at a distance of \( r_{12} \):

\[
g_1 = G m_1 / r_{12}^2
\]

\( g_2 \) since \( m_1 < m_2 \)

Gravitational force on the earth’s surface

Find the gravitational field \( g \) on the earth’s surface.

\[
g = G m_E / R_E^2
\]

\[
= (6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2) \left( \frac{5.98 \times 10^{24} \text{ kg}}{(6.38 \times 10^8 \text{ m})^2} \right)
\]

\[
= (6.67 \times 10^{-11} \text{ N} / \text{m}^2) \left( \frac{(5.98 \times 10^{24})}{(6.38)^2 \times 10^{16}} \right)
\]

\[
= 9.80 \text{ m/s}^2
\]

Earth and Moon

Using the fact that the gravitational field at the surface of the Earth is about six times larger than that at the surface of the Moon, and the fact that the Earth’s radius is about four times the Moon’s radius, determine how the mass of the Earth compares to the mass of the Moon.

\[
\frac{g_1}{g_2} = \left( \frac{r_2}{r_1} \right)^2 \Rightarrow \frac{m_2}{m_1} = \left( \frac{r_2}{r_1} \right)^2 \left( \frac{g_1}{g_2} \right) = \left( \frac{1}{4} \right)^2 \left( \frac{1}{6} \right) = \frac{1}{96}
\]

So the mass of the moon is 1/96 times of that of the earth.

Three masses on a straight line

(a) Three masses, of mass \( 2M, M, \) and \( 3M \) are equally spaced along a line, as shown. The only forces each mass experiences are the forces of gravity from the other two masses.

(i) Which mass experiences the largest magnitude net force?

1. \( 2M \)
2. \( M \)
3. \( 3M \)
4. Equal for all three
5. Net force magnitude on \( 2M \) is equal to that on \( 3M \) but bigger than that on \( M \)

(ii) What’s the magnitude of the net force experienced by mass \( 2M \)?

Solution

We can just add the forces from the other two objects. The net force on the \( 2M \) object is directed right with a magnitude of:

\[
F_{2M} = \frac{G M \cdot 2M}{R^2} + \frac{G \cdot 3M \cdot 2M}{(2R)^2}
\]

\[
= \frac{4GMM}{2R} + \frac{3GMM}{2R} = \frac{7GMM}{2R^2}
\]
A triangle of masses

Three point objects, 1 through 3 with identical mass, are placed at the corners of an equilateral triangle.

In what direction is the net gravitational field at point A, halfway between objects 2 and 3?

Net gravitational field at point A

The net gravitational field at point A comes from three sources, objects 1, 2, and 3. The diagram below shows the corresponding three gravitational fields, \( g_1, g_2 \) and \( g_3 \). Obviously, \( g_2 \) and \( g_3 \) cancel. The net field at A is due only to \( g_1 \), which points up.

\[
g_1 = \frac{Gm}{r_{1A}^2} = \frac{1}{1/r_{1A}^2}
\]

where \( r_{1A} \) is the distance between point A and object 1, \( m \) is the mass of the objects.

Net gravitational force at point A

Find the net gravitational force experienced by an object of mass 2M at point A in terms of the mass of the three source objects \( m \) and the length of each side of the equilateral triangle, \( L \).

Solution

From the above discussion, the net gravitational field is \( g_1 \), which is \( \frac{Gm}{r_{1A}^2} \). But \( r_{1A} = L\sin60^\circ = \left(\frac{\sqrt{3}}{2}\right)L \). So,

\[
g_1 = \frac{2Gm}{3L^2}
\]

The net gravitational force, \( F \), experienced by an object with mass 2M is 2M times the field at that point. So,

\[
F = (2M)g_1 = 4GmM/(3L^2).
\]

Velocity of Orbiting Satellites

There is only one speed, \( v \), that a satellite can have if the satellite is to remain in a circular orbit with radius, \( r \). What is the relation between \( v \) and \( r \)? What is \( v \) at \( r = 10R_E \)?

Solution:

For the satellite to remain in a circular orbit with radius \( r \), the acceleration of the satellite must equal \( a = v^2/r \); for otherwise the unbalanced force, \( F_{\text{net}} - ma \), will cause the satellite to move away from the (circular) orbit.

\[
m\frac{v^2}{r} = mg = \frac{GMm}{r^2}
\]

\[\Rightarrow v = \sqrt{\frac{GM}{r}}
\]

Three masses on a straight line

(iii) Which of the following changes would cause the magnitude of the force experienced by the 2M object to increase by a factor of 4? Select all that apply.

- [ ] double the mass of all three objects
- [ ] change the mass of the 2M object to 8M, without changing the mass of the other objects
- [ ] double \( R \)
- [ ] Move the system to a parallel universe where the value of the universal gravitational constant is four times larger than its value in our universe

Velocity of Orbiting Satellites

Substitute \( r = 10R_E \) to find \( v \) in the orbit:

\[
v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-2} \times 5.97 \times 10^{24} \text{ kg}}{10 \times 6.38 \times 10^6 \text{ m}}}
\]

\[
= \frac{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-2} \times 5.97 \times 10^{24} \text{ kg}}{6.38 \times 10^6 \text{ m}}
\]

\[= 2.50 \text{ km/s}
\]

This is faster than the speed of any plane ever flown on earth (which is ~2 km/s)!
In many applications of satellites such as digital satellite system television, it is desirable that the motion of the satellite follows a circular orbit and be synchronized with the earth’s self rotation (so that the satellite is always at the same location above the earth’s surface). This requires that the period, \( \tau \), of the satellite be exactly one day, i.e., \( 8.64 \times 10^4 \) s. What is the height, \( H \), of the satellite above the earth’s surface?

The Orbital Radius for Synchronous satellites

\[
\frac{2\pi r}{r} = \sqrt{\frac{GM}{r}} \Rightarrow r = \frac{GM}{2\pi} \Rightarrow r = 4.22 \times 10^7 \text{ m}
\]

So, \( H = r - R_E = 3.58 \times 10^7 \) m

General Orbital Motions of satellites

1. They often trace out an ellipse. Therefore, the gravitational force is not always perpendicular to the satellite’s velocity.
2. \( K + U \) is conserved throughout the orbit.
3. Angular momentum is conserved throughout the orbit.
4. Linear momentum is not conserved since there is gravitational force.
5. The orbit period does not depend on the mass of the satellite.
6. They obey the Kepler’s 2nd Law. That is, equal areas of the orbit are swept out in equal time intervals.

Gravitational potential energy

The gravitational interaction or potential energy of two objects with masses \( m \) and \( M \) and separation \( r \) is:

\[
U_{ij} = -\frac{GmM}{r}
\]

The negative sign tells us that the interaction is attractive. Note that with this equation the potential energy is defined to be zero when \( r = \infty \).

What matters is the change in gravitational potential energy. For small changes in height at the Earth’s surface, i.e., from \( r = R_E \) to \( r = R_E + h \), the equation above gives the same change in potential energy as \( mgh \), where \( g = \frac{GM}{R_E^2} \) as found before.

Four objects in a square

Four objects of equal mass, \( m \), are placed at the corners of a square that measures \( L \) on each side. How much gravitational potential energy is associated with this configuration of masses?

The gravitational PE is the sum of all the six \( U_{ij} \)’s, which is \((4+4\sqrt{2})\frac{Gm^2}{L}\).

Convince yourself that the answer is independent of the order by which we bring the charges in.
Gravitational potential energy of a distribution of masses

In general, the gravitational potential energy, \( U \), of a distribution of masses comprising masses \( m_i \) (where \( i = 1, 2, 3, \ldots, N \)) is:

\[
U = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \ldots + \frac{m_1 m_N}{r_{1N}} + \frac{m_2 m_3}{r_{23}} + \frac{m_2 m_4}{r_{24}} + \ldots + \frac{m_2 m_N}{r_{2N}} + \ldots + \frac{m_N-1 m_N}{r_{N-1,N}}\right),
\]

where \( r_{ij} \) is the distance between \( m_i \) and \( m_j \) (\( i, j = 1, 2, 3, \ldots, N \)).

Escape speed

How fast would you have to throw an object so it never came back down? Ignore air resistance. Find the escape speed - the minimum speed required to escape from a planet's gravitational pull.

Solution:

Approach to use: Forces are hard to work with here because the size of the force changes as the object gets farther away. Energy is easier to work with in this case.

Conservation of energy:

\[ U_i + K_f + W_{nc} = U_i + K_f \]

Which terms can we cross out immediately?

1. Assume no resistive forces, so \( W_{nc} = 0 \).
2. Assume that the object barely makes it to infinity, so both \( U_f \) and \( K_f \) are zero.

This leaves: \( U_i + K_f = 0 \)

\[ -Gm_i M / R_E + \frac{1}{2} m v_{\text{escape}}^2 = 0 \]

\[ v_{\text{escape}}^2 = 2 \frac{Gm_i M}{R_E^2} \]

\[ v_{\text{escape}} = \sqrt{\frac{2 G m_i M}{R_E^2}} = 9.8 \text{ m/s}^2 \]

\[ v_{\text{escape}} = \sqrt{\frac{2Gm_i M}{R_E^2}} = 9.8 \text{ m/s}^2 \]

\[ v_{\text{escape}} = \sqrt{\frac{2 \cdot 9.8 \text{ m/s}^2 \cdot 6.38 \times 10^6 \text{m}}{R_E^2}} = 11.2 \text{ km/s} \]