1. (Jackson 11.3)
Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with a velocity

\[ v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \]

This is an alternate way to derive the parallel-velocity addition law.

Let

\[ \beta \equiv \frac{v_1}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}} \]

\[ \beta' \equiv \frac{v_2}{c}, \quad \gamma' \equiv \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}} \]

The Lorentz transformations are

\[ \Lambda = \begin{pmatrix} \gamma & \beta \gamma & 0 & 0 \\ \beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda' = \begin{pmatrix} \gamma' & \beta' \gamma' & 0 & 0 \\ \beta' \gamma' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

so

\[ \Lambda \Lambda' = \begin{pmatrix} \gamma' (1 + \beta \beta') & \gamma' (\beta + \beta') & 0 & 0 \\ \gamma' (\beta + \beta') & \gamma' (1 + \beta \beta') & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
Let

\[ \Lambda'' = \Lambda\Lambda' = \begin{pmatrix} \gamma'' & \beta'\gamma' & 0 & 0 \\ \beta'' & \gamma'' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

so

\[ \gamma'' = \gamma\gamma'(1 + \beta\beta') \]

and

\[ \beta''\gamma'' = \gamma\gamma'(\beta + \beta') \]

which gives

\[ \beta'' = \frac{\beta + \beta'}{1 + \beta\beta'} \]

and

\[ \gamma'' = \gamma\gamma'(1 + \beta\beta') = \frac{1 + \beta\beta'}{\sqrt{1 - \beta^2}\sqrt{1 - \beta'^2}} = \frac{1}{\sqrt{1 - \left(\frac{\beta + \beta'}{1 + \beta\beta'}\right)^2}} = \frac{1}{\sqrt{1 - \beta''^2}} \]

This is the velocity addition rule.
2. (Jackson 11.4)
A possible clock is shown in the figure. It consists of a flashtube $F$ and a photocell $P$
shielded so that each views only the mirror $M$, located a distance $d$ away, and mounted
rigidly with respect to the flashtube-photocell assembly. The electronic innards of the box
are such that when the photocell responds to a light flash from the mirror, the flashtube is
triggered with a negligible delay and emits a short flash toward the mirror. The clock thus
“ticks” once every $(2d/c)$ seconds when at rest.

(a) Suppose that the clock moves with a uniform velocity $v$, perpendicular to the line
from $PF$ to $M$, relative to an observer. Using the second postulate of relativity, show by
explicit geometrical or algebraic construction that the observer sees the relativistic time
dilation as the clock moves by.

Let $K$ be the frame in which the clock is at rest. One tick takes
\[
\Delta t = \frac{2d}{c}
\]

In frame $K'$, as the clock moves by, the light beam must travel a
greater distance (each way) in the time $\Delta t'$. Therefore, the vertical
component of velocity is less than $c$.

\[
\Delta t' = \frac{2d'}{\sqrt{c^2 - v^2}} = \frac{2d}{c} \frac{1}{\sqrt{c^2 - v^2}} = \gamma \Delta t
\]

The time interval is longer in the frame where the clock is moving.

(b) Suppose that the clock moves with a velocity $v$ parallel to the line from $PF$ to $M$.
Verify here, too, the clock is observed to tick more slowly, by the same time dilation factor.
Along one path, say $FM$, the light travels a longer path (because the mirror moves away), but along the other path, say $MP$, the light travels a shorter path (because the flasher is approaching).

Let $t'_1$ be the time taken for light to arrive at the mirror.

$$ct'_1 = d' + vt'_1$$

This gives

$$t'_1 = \frac{d'}{c - v}$$

Let $t'_2$ be the time taken for light to arrive back at the flasher box.

$$ct'_2 = d' - vt'_1$$

This gives

$$t'_2 = \frac{d'}{c + v}$$

The time interval in $K'$ is

$$\Delta t' = t'_1 + t'_2 = \frac{d'}{c + v} = \frac{2cd'}{c^2 - v^2}$$

The length $d'$ is contracted,

$$d' = d\sqrt{1 - \frac{v^2}{c^2}}$$

which gives

$$\Delta t' = \frac{2cd\sqrt{1 - \frac{v^2}{c^2}}}{c^2\left(1 - \frac{v^2}{c^2}\right)} = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$
Again, the time interval is longer in the frame where the clock is moving.
3. (Jackson 11.6)
Assume that a rocket ship leaves the earth in the year 2100. One of a set of twins born in 2080 remains on earth; the other rides in the rocket. The rocket ship is so constructed that it has an acceleration \( g \) in its own rest frame (this makes the occupants feel at home). It accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. The twin in the rocket is 40 years old.
(a) What year is it on earth?

Let \( K \) be the earth frame. In this frame the rocket has a speed \( v(t) \).
Let \( K' \) be a frame that moves with the rocket. Each segment of the flight has a duration of 5 years in this frame.

The trick to this problem is that we can do a Lorentz transformation at each INSTANT.

Consider the time \( t \) when the speed of the rocket is \( v \) in frame \( K \).
In frame \( K' \),
\[
v'(t') = 0
\]
Consider an *infinitesimal* change in speed,
\[
\Delta v' = g \Delta t'
\]
In frame \( K \), the new rocket velocity is (by velocity addition)
\[
v_{\text{new}} = \frac{\Delta v' + v}{1 + \frac{v \Delta v'}{c^2}} \approx \left( \Delta v' + v \right) \left( 1 - \frac{v \Delta v'}{c^2} \right)
\]
\[
= \Delta v' + v - \frac{v^2 \Delta v'}{c^2} + \text{order(} \Delta v'^2 \text{)}
\]
therefore,
\[ v_{new} - v = \Delta v = \left(1 - \frac{v^2}{c^2}\right)\Delta v' \]

or

\[ dv = \left(1 - \frac{v^2}{c^2}\right)dv' \]

and

\[ \frac{dv}{dt'} = \left(1 - \frac{v^2}{c^2}\right)dv' = \left(1 - \frac{v^2}{c^2}\right)g \]

This gives

\[ \frac{dv}{1 - \frac{v^2}{c^2}} = gdt' \]

Integrate to get

\[ t' = \frac{1}{g} \int \frac{1}{1 - \frac{v^2}{c^2}} dv = \frac{c}{g} \tanh^{-1}\left(\frac{v}{c}\right) \]

or

\[ v = c \tanh\left(\frac{gt'}{c}\right) \]

This is the speed of the spacecraft observed on earth as a function of spacecraft time as it accelerates from rest.

Time measured on earth is related to spacecraft time by
\[
\frac{dt}{dt'} = \frac{dt'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{dt'}{\sqrt{1 - \tanh^2\left(\frac{gt'}{c}\right)}} = \cosh\left(\frac{gt'}{c}\right) dt'
\]

or

\[
t = \int dt' \cosh\left(\frac{gt'}{c}\right) = \frac{g}{c} \sinh\left(\frac{gt'}{c}\right)
\]

Now plug in, for \(t' = 5\) years, we get \(t = 84\) years.

For all 4 segments, the amount of time passing on earth is 336 years. The year is 2436.

(b) How far away from the earth did the rocket ship travel?

\[
\frac{dx}{dt} = v = c \tanh\left(\frac{gt'}{c}\right) = c \frac{\sinh\left(\frac{gt'}{c}\right)}{\cosh\left(\frac{gt'}{c}\right)}
\]

\[
= c \frac{\sinh\left(\frac{gt'}{c}\right)}{\sqrt{1 + \sinh^2\left(\frac{gt'}{c}\right)}} = \frac{gt}{\sqrt{1 + \frac{g^2 t^2}{c^2}}}
\]

so

\[
x = \int dt \frac{gt}{\sqrt{1 + \frac{g^2 t^2}{c^2}}} = \frac{c^2}{g} \left( \sqrt{1 + \frac{g^2 t^2}{c^2}} - 1 \right)
\]
which for $t = 5$ years gives $x = 7.95 \times 10^{17}$ m.

The rocket travels twice that distance away from the earth, or $1.59 \times 10^{18}$ m.