PY252 Problem Set 2 Solutions

As always, I urge you to perform all calculations symbolically to their completion before substituting any numerical values for any quantities. You will get a better understanding of the physics involved, and will be able to apply the same work to all problems with different numerical input values but the same physics.

Problem 1 (Princeton 13-II)
See text for solution

Problem 2 (Princeton 13-III)
See text for solution

Problem 3 (Princeton 13-V)
See text for solution

Problem 4 (Purcell 1-1)
In the domain of elementary particles, a natural unit of mass is the mass of a nucleon, that is, a proton or a neutron, the basic massive building blocks of ordinary matter. Given the nucleon mass as $1.6 \times 10^{-24}$ g and the gravitational constant $G$ as $6.7 \times 10^{-8}$ cm$^3$/gcm, compare the gravitational attraction of two protons with their electrostatic repulsion. This shows why we call gravitation a very weak force. The distance between the two protons in the helium nucleus could be at one instant as much as $10^{-13}$ cm. How large is the force of electrical repulsion between two protons at that distance? Express it in Newtons, and in pounds. Even stronger is the nuclear force that acts between any pair of hadrons (including neutrons and protons) when they are that close together.

The gravitation force between two masses, you will hopefully remember, is given by:

$$F = \frac{-G m_1 m_2}{r^2} \hat{r}$$

while the electrical force is given by:

$$F = \frac{q_1 q_2}{r^2} \hat{r}$$

Thus, the ratio of the magnitudes of the two types of forces is

$$R = \frac{G m_1 m_2}{q_1 q_2}$$

For protons, this value is numerically:

$$R = 6.7 \times 10^{-8} \frac{cm^3}{gs^2} \left( \frac{1.6 \times 10^{-24} g}{4.8 \times 10^{-10} \text{esu}} \right)^2 = 7.4 \times 10^{-37}$$

For a separation of $r = 10^{-13}$ cm, the electrical force between the protons is:

$$F = \frac{e^2}{r^2} = 2.3 \times 10^6 \text{dynes} = 23 N = 5.2 \text{lb}$$
Five pounds! Wow!¹

**Problem 5 (Purcell 1-3)**

Two volley balls, mass 0.3 kg each, tethered by nylon strings and charged with an electrostatic generator, hang as shown in the diagram below. What is the charge on each in coulombs, assuming the charges are equal? (Remember: the weight of a 1 kg mass on earth is 9.8 N, just as the weight of a 1 g mass is 980 dynes.)

![Diagram of two equally charged balls](image1)

Figure 1: Two equally charged, equal mass spheres. What is the charge?

We'll first draw a free body diagram (Remember those from last term? You should!) of the situation above. Again, I will do this symbolically, and substitute the numbers later:

![Free body diagram](image2)

Figure 2: Free body diagram

The gravitational force of course is given by:

\[ F_g = Mg \]

¹This is one of those times when you should realize just how cool physics is.
While the electrical repulsion will be given by:

$$F_q = \frac{q^2}{4\pi\varepsilon_0 d^2}$$

(since the data is given in SI units, I will use the SI formulation of electrostatics here). The ratio of the forces will be the same as the ratio of the lengths $h$ and $r = d/2$:

$$\frac{F_q}{F_g} = \frac{r}{h}$$

Thus, the charge $q$ will be given by:

$$q = \left(8\pi\varepsilon_0 mghd\right)^{\frac{1}{2}}$$

For this problem, we can substitute the numerical values into this result, and we will obtain the charge:

$$q = 2.86 \times 10^{-5} \text{ C}$$

Problem 6 (Purcell 1-4)

At each corner of a square is a particle with charge $q$. Fixed at the center of the square is a point charge of opposite sign, of magnitude $Q$. What value must $Q$ have to make the total force on each of the four particles zero? With $Q$ set at that value, the system, in the absence of other forces, is in equilibrium. Do you think the equilibrium is stable?

![Figure 3: Charge layout for Problem 5.](image)

From the diagram, we see that the system has a symmetry, such that we only need to look at one of the corner charges to determine the value of $Q$. In particular, there are four forces on one of the charges; we’ll look at the one in the upper right corner. In fact, given the symmetry, we only need to look at the $x$ (or $y$) components of the forces, since they will be equal. We’ll call the $x$-components of the four forces $F_Q$, $F_{LL}$, $F_{UL}$, and $F_{LR}$ (for lower left, upper left, and lower right respectively). The forces are given by:

$$F_Q = \frac{qQ}{\left(a/\sqrt{2}\right)^2 \sqrt{2}} \quad F_{LL} = 0$$

$$F_{UL} = \frac{q^2}{\left(\sqrt{2}a\right)^2 \sqrt{2}}$$

$$F_{UL} = \frac{q^2}{a^2}$$
Now, we sum these contributions, and insist that they sum to zero:

\[ F = \frac{q^2}{a^2} + \frac{q^2}{2\sqrt{2}a^2} + \frac{4qQ}{2\sqrt{2}a^2} \rightarrow Q = -\left(\frac{1 + 2\sqrt{2}}{4}\right)q \]

Is this situation stable? Certainly not! Any displacement of the charge \( Q \) from the plane of the four charges \( q \) will result in the charge being further displaced from the plane. If you so desire, you can apply the methods you learned in PY251 to study the stability of orbits to this question and obtain a more quantitative “no” result.

**Problem 7 (Purcell 1-7)**

*Find a geometrical arrangement of one proton and two electrons such that the potential energy of the system is exactly zero. How many such arrangements are there with the three particles on the same straight line?*

The potential energy of a configuration of two electrons and one proton is given (from Equation 1-7) by:

\[ U = \frac{q_e q_2}{r_{ee}} + \frac{q_p q_1}{r_{pe_1}} + \frac{q_p q_2}{r_{pe_2}} \]

Since we want \( U = 0 \) and we know that \( q_p = -q_e = e \), you should be able to see that this requires:

\[ \frac{1}{r_{pe_1}} + \frac{1}{r_{pe_2}} - \frac{1}{r_{e_1 e_2}} = 0 \]

There are an infinite number of configurations which satisfy this mathematical relation, yet are still physical; this is easy to prove, as this relation has three unknowns, so the system is overspecified. You, however, are only asked to find one where the particles are arranged collinearly. There are only two classes of arrangements that are possible on a line: either the proton is between the two electrons, or the proton is on one end of the chain. First, consider the proton centered arrangement. If the separations between the proton and the electrons are \( a \) and \( b \) respectively, then the potential energy will be:

\[ U \propto \frac{1}{a} + \frac{1}{b} - \frac{1}{a + b} \]

which can be solved for \( U = 0 \) by the solutions of:

\[ a^2 + ab + b^2 = 0 \]

which can never be zero with any physical, non-zero separations.\(^2\) In the second configuration, with the proton on the end, if we define the electron separation as \( r \) and the distance between the proton and the closer electron as \( R \), then it follows that:

\[ U \propto \frac{1}{R} + \frac{1}{R + r} - \frac{1}{r} = 0 \]

we then crank the handle on the algebra machine to find:

\[ r^2 + Rr - R^2 = 0 \]

Since we have two unknowns and only one equation, we can take one of the unknowns as a free parameter (I’ll choose \( R \)), and solve the quadratic:

\[ r = \frac{-R \pm \sqrt{R^2 + 4R^2}}{2} \]

taking the only physical solution, we find an infinite set of solutions:

\[ r = \frac{\sqrt{5} - 1}{2}R \]

\(^2\)If we solve for \( a \), we obtain

\[ a = \frac{-b \pm \sqrt{-3b^2}}{2} = \frac{-1 \pm \sqrt{3}}{2}b \]
Problem 8 (Purcell 1-9)

A spherical volume of radius a is filled with charge of uniform density ρ. We want to know the potential energy U of this sphere of charge, that is, the work done in assembling it. Calculate it by building the sphere up layer by layer, making use of the fact that the field outside a spherical distribution of charge is the same as if all the charge were at the center. Express the result in terms of the total charge Q in the sphere.

We want to build up this sphere of charge Q by adding slowly layer upon layer from zero. Assume then that we are already partway through this procedure, and we have a sphere of radius q(r < a). The next step is to bring in thin spherical shell from infinity to r. If the shell is thin enough, we can ignore the effect of the self interaction of the shell on the potential energy. If we can do so, it is certainly the case that bringing the shell in a small wedge at a time is just as valid. Then, the potential energy of this sphere plus wedge configuration will be:

\[
dU(r) = \frac{q(r) dq(r)}{r}
\]

with

\[
q(r) = \rho \frac{4\pi}{3} r^3
\]

and:

\[
dq(r) = \rho (r d\phi)(r \sin \theta d\theta) dr
\]

where the first term is the density of the charge, the second term is the “length” of the wedge, the third term the “width” of the wedge, and the final term the “depth” of the wedge. We can sum over all these added charges by integrating this expression over the entire volume of the sphere:

\[
U(a) = \int dU(r) = \rho \frac{4\pi}{3} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^a r^4 dr
\]

when we perform the angular integrals, we obtain 4π. Performing the radial integral, we obtain \(\frac{Q^2}{a}\). Thus, we have obtained:

\[
U(a) = \left( \frac{4\pi}{3} a^3 \right)^2 \frac{3}{5a} = \frac{3Q^2}{5a}
\]

A second method of solution (which you are not asked to pursue, but which I find much more compelling, not to mention easier) would be to follow the result of equation 38:

\[
U = \frac{1}{8\pi} \int_{\text{Entire Field}} E^2 dV
\]

Since we can use Gauss’ Law to find the field:

\[
E(r) = \frac{Q}{r^2}; \quad r > a
\]

\[
E(r) = \frac{1}{r^2} Q \frac{r^3}{a^3} = \frac{Qr}{a^3}; \quad r < a
\]

Then, we can perform the integration:

\[
U = \frac{1}{8\pi} \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^\infty E^2(r) r^2 dr
\]

The angular integrations are trivial and total 4π. The radial integral can be done easily as well:

\[
\int_0^\infty E^2(r) r^2 dr = \int_0^a \frac{Q^2 r^4}{a^6} dr + \int_a^\infty \frac{Q^2}{r^4} dr = \frac{Q^2}{5a} + \frac{Q^2}{a}
\]

Combining these parts we find the value of the potential energy:

\[
U = \frac{3Q^2}{5a}
\]
which is, of course, the same answer that we obtained earlier.

Problem 9 (Purcell 1-10)

At the beginning of the century, the idea that the rest mass of electron might have a purely electrical origin was very attractive, especially when the equivalence of energy and mass was revealed by special relativity. Imagine the electron as a ball of charge, of constant volume density out to some maximum radius \( r_0 \). Using the result of the previous problem, set the potential energy of this system equal to \( mc^2 \) and see what you get for \( r_0 \). One defect of the model is rather obvious: Nothing is provided to hold the charge together!

Starting with the result we found in the last problem:

\[
U = \frac{3}{5} \frac{Q^2}{r_0} = mc^2
\]

Solving for \( a \), we can find the radius trivially enough:

\[
r_0 = \frac{3e^2}{5mc^2} = 1.69 \times 10^{-13} \text{cm}
\]