PY252 Problem Set 11 Solutions

As always, I urge you to perform all calculations symbolically to their completion before substituting any numerical values for any quantities. You will get a better understanding of the physics involved, and will be able to apply the same work to all problems with different numerical input values but the same physics.

Problem 1 (Purcell 9-1)

If the electric field in free space is \( \vec{E} = E_0 (\hat{x} + \hat{y}) \sin \left( \frac{2\pi}{\lambda} (z + ct) \right) \), with \( E_0 = 2 \text{ statvolt/cm} \), the magnetic field, not including any static magnetic field, must be what?

Since the field is travelling in the \(-z\) direction (since the time variation goes like \( z + ct \)), the direction of travel is given by the direction of \( \vec{E} \times \vec{B} \), and \( |\vec{E}| = |\vec{B}| \), we must have the result that

\[
\vec{B} = E_0 (\hat{x} - \hat{y}) \sin \left( \frac{2\pi}{\lambda} (z + ct) \right)
\]

Problem 2 (Purcell 9-3)

A free proton was at rest at the origin before the wave described by

\[
\vec{E} = \frac{5\hat{y}}{1 + (x + ct)^2}
\]

\[
\vec{B} = \frac{-5\hat{x}}{1 + (x + ct)^2}
\]

came past. Where would you expect to find the proton at time \( t = 1 \text{ \mu s} \)? The pulse amplitude is in statvolt/cm, and the proton mass is \( m_p = 1.6 \times 10^{-24} \text{ gm} \). Hint: Since the duration of the pulse is only a few nanoseconds, you can neglect the displacement of the proton during the passage of the pulse. Also, if the velocity of the proton is not too large, you may ignore the effect of the magnetic field on its motion. The first thing to calculate is the momentum acquired by the proton during the pulse.

Since we are going to be ignoring the magnetic component of the force on this proton during the passage of the wave pulse, we need only calculate the force on the proton, and integrate to find the displacement. The force on the proton at the origin will be

\[
F_y = \frac{dp_y}{dt} = eE_y
\]

which we can integrate to find the momentum \( p_y \)

\[
p_y = \int_{-\infty}^{\infty} e \frac{5}{1 + (ct)^2} = \frac{5e}{c} \int_{-\infty}^{\infty} \frac{dt'}{1 + t'^2}
\]

How do we perform the integration? As usual, this integral is one of the elementary integrals that you should have learned at some point

\[
\int \frac{dx}{1 + x^2} = \tan^{-1} x - C
\]

so, when we integrate, we obtain

\[
p_y = \frac{5e\pi}{c}
\]

When we realize that the majority of this acceleration occurs over a very small region of time surrounding the center of the pulse, then we see that we can approximate the result after a time \( T = 1 \text{ \mu s} \) by the full result (perform
both computations, and compare the results if you don’t believe me!). Furthermore, this is such a small acceleration that the final velocity of the proton is non-relativistic (again, as we have done many times before, let us make the assumption that this is correct, calculate the consequences, and then check that our result is consistent with our approximation). In this case,

$$v_y = \frac{p_y}{m_p}$$

Substituting the values in this problem, we’ll find that after 1 ms,

$$d_y = v_y t = \frac{5e\pi t}{m_pc} = 0.15 \text{ cm}$$

**Problem 3 (Purcell 9-5)**

*Here is a particular electromagnetic field in free space:*

$$E'_x = 0 \quad E'_y = E_0 \sin(kx + \omega t) \quad E'_z = 0$$

$$B'_x = 0 \quad B'_y = 0 \quad B'_z = -E_0 \sin(kx + \omega t)$$

1. Show that this field can satisfy Maxwell’s equations if ω and k are related in a certain way.

2. Suppose $\omega = 10^{10} s^{-1}$ and $E_0 = 0.05 \text{ statvolt/cm}$. What is the wavelength in cm? What is the energy density in ergs/cm$^2$, averaged over a large region? From this calculate the power density, the energy flow in ergs/cm$^2$s.

1. Checking Maxwell’s source free equations is straight-forward; just plug into the equations and check the result. Maxwell’s source free equations read

\[
\nabla \cdot \vec{E} = 0 \\
\nabla \cdot \vec{B} = 0 \\
\n\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\
\n\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}
\]

What are the results of these operations? I’ll just quote them here

$$\nabla \cdot \vec{E}' = \nabla \cdot \vec{B}' = 0$$

$$\nabla \times \vec{E}' = \hat{x}kE_0 \cos(kx + \omega t)$$

$$\nabla \times \vec{B}' = \hat{y}kE_0 \cos(kx + \omega t)$$

$$\frac{\partial \vec{E}'}{\partial t} = \hat{x}\omega E_0 \cos(kx + \omega t)$$

$$\frac{\partial \vec{B}'}{\partial t} = -\hat{y}\omega E_0 \cos(kx + \omega t)$$

Substituting these into the Maxwell equations, we see that, for this choice of $\vec{E}'$ and $\vec{B}'$, the Maxwell equations are satisfied, if and only if $\omega = k\varepsilon$.

2. The energy density of an electromagnetic wave at a point $\vec{x}$ is given by

$$U_{EM}(\vec{x}) = \frac{E(\vec{x})^2}{8\pi} + \frac{B(\vec{x})^2}{8\pi}$$

The energy $T_V$ stored in the electromagnetic fields in a given volume $V$ is given by integrating over the volume

$$T_V = \int_V U_{EM}(\vec{x}) dV$$
The average energy density is just \( \bar{U} = T_V/V \). If we integrate over a box one wave-length in linear dimension, then the average energy density will be just half the peak energy density (the integral of a sinusoid over a full period is \( 1/2 \)). Furthermore, the average power density is

\[
S = c \frac{T_V}{V} = \bar{U}
\]

With the numerical values given in this problem, we find the following values

\[
\bar{U} = 0.99 \times 10^{-4}\text{erg cm}^{-2}
\]

\[
S = 3.0 \times 10^9\text{erg cm}^3\text{s}^{-1}
\]

Problem 4 (Purcell 10-9)

The magnetic field inside a circular, discharging capacitor, of radius \( b \) and plate separation \( s \), can in principle be calculated by summing the contributions from all elements of the conduction current. That might be a long job. If we can assume symmetry about this axis, it is very much easier to find the field \( \vec{B} \) at a point using the integral law

\[
\int_C \vec{B} \cdot d\vec{s} = \frac{1}{c} \int_S \left( \frac{\partial \vec{E}}{\partial t} + 4\pi \vec{J} \right) \cdot d\vec{A}
\]

applied to a circular path through the point. We need only know the total current enclosed by this path. Use this to find the field at point \( P \) applied to a circular path through the point. We need only know the total current enclosed by this path. Use words to find the field which is midway between the capacitor plates; a distance \( r \) from the axis of symmetry. (Compare this with the calculation of the induced electric field \( \vec{E} \), in the example in the text.)

We'll make the assumption that the plates are large enough (\( s \ll b \)) that the current (and hence, the electric field lines) travels directly from one plate to the other, perpendicular to the plates. If we are looking at a small enough time period, the electric field will be constant in time, so we can ignore the time derivative; in this limit, the integral on the right hand side of the equation gives the total current travelling along the wire, \( I \), and this current will be uniformly distributed over the area of the plates (again, only when we are far from the edges; we will ignore the small error here). By symmetry, the magnitude of the field \( \vec{B} \) will be constant at a given radius \( r \), and this problem reduces to the calculation of the magnetic field inside a wire carrying a constant current density (thereby turning a hard problem into an easy one, that is, one that you've already solved!). Thus,

\[
2\pi r B(r) = \frac{4\pi I}{c} \frac{r^2}{b^2} \rightarrow B(r) = \frac{2Ir}{cb^2}
\]

Problem 5 (Purcell 9-13)

Starting from the Lorentz field transformations, show that the scalar quantity

\[
E^2 - B^2
\]

is a Lorentz scalar quantity, that is invariant under the transformation. In other words, show that

\[
E'^2 - B'^2 = E^2 - B^2
\]

You can do this using only vector algebra, without writing out the components of anything. The resolution into parallel and perpendicular vectors is convenient for this, since \( \vec{E}_\parallel \cdot \vec{E}_\perp = 0 \) and \( \vec{B}_\parallel \times \vec{E}_\parallel = 0 \).

We'll find the answer by plunging straight in, and cranking the handle on the vector algebra machine. First, let us break the expression in the primed frame into vector components parallel and perpendicular to the boost vector \( \vec{\beta} \)

\[
E^2 - B^2 = \vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B} = \left( \vec{E}_\parallel + \vec{E}_\perp \right) \cdot \left( \vec{E}_\parallel + \vec{E}_\perp \right) - \left( \vec{B}_\parallel + \vec{B}_\perp \right) \cdot \left( \vec{B}_\parallel + \vec{B}_\perp \right)
\]
We can further expand the dot products on the extreme right by distributing; since the parallel and perpendicular components are clearly orthogonal, their dot product gives zero. Now, we can replace the variables in the primed frame with their values in the unprimed frame

\[
\begin{align*}
\vec{E}_\parallel &= \vec{E}_\parallel \\
\vec{B}_\parallel &= \vec{B}_\parallel \\
\vec{E}_\perp &= \gamma (\vec{E}_\perp + \vec{\beta} \times \vec{B}_\perp) \\
\vec{B}_\perp &= \gamma (\vec{B}_\perp - \vec{\beta} \times \vec{E}_\perp)
\end{align*}
\]

Then the parallel components will give us

\[
E'^2 - B'^2 = \vec{E}_\parallel \cdot \vec{E}_\parallel - \vec{B}_\parallel \cdot \vec{B}_\parallel = \vec{E}_\parallel \cdot \vec{E}_\parallel - \vec{B}_\parallel \cdot \vec{B}_\parallel
\]

and the perpendicular components

\[
\vec{E}_\perp \cdot \vec{E}_\perp - \vec{B}_\perp \cdot \vec{B}_\perp = \gamma^2 (\vec{E}_\perp + \vec{\beta} \times \vec{B}_\perp) \cdot (\vec{E}_\perp + \vec{\beta} \times \vec{B}_\perp) - \gamma^2 (\vec{B}_\perp - \vec{\beta} \times \vec{E}_\perp) \cdot (\vec{B}_\perp - \vec{\beta} \times \vec{E}_\perp)
\]

This horrible expression can be simplified by distributing and noting that \(\vec{E}_\perp \cdot \vec{\beta} = \vec{B}_\perp \cdot \vec{\beta} = 0\), and further \((\vec{\beta} \times \vec{E}_\perp)^2 = \beta^2 \vec{E}_\perp^2\), with the same result for the \(\vec{B}\) field. Finally, the scalar triple products can be simplified since \(\vec{E}_\perp \cdot (\vec{\beta} \times \vec{B}_\perp) = -\vec{B}_\perp \cdot (\vec{\beta} \times \vec{E}_\perp)\). Thus, we obtain

\[
\vec{E}_\perp \cdot \vec{E}_\perp - \vec{B}_\perp \cdot \vec{B}_\perp = \gamma^2 [E_\perp^2 (1 - \beta^2) - B_\perp^2 (1 - \beta^2)] = E_\perp^2 - B_\perp^2
\]

Combining the two results

\[
E'^2 - B'^2 = E_\parallel^2 - B_\parallel^2 + E_\perp^2 - B_\perp^2 = E^2 - B^2
\]

Thus, we’ve shown that the quantity \(E^2 - B^2\) is a Lorentz invariant scalar quantity.