9-9

At what displacement of an object undergoing simple harmonic motion is the magnitude greatest for the...

(a) velocity?

The velocity is greatest at $x = 0$, the equilibrium position. At any other position, there is both kinetic energy and spring energy, but at the equilibrium, all of the energy is kinetic. Maximum kinetic energy $\rightarrow$ maximum velocity

(b) acceleration?

Newton’s law applied to a spring tells us: $F = ma = -kx \rightarrow a = -\frac{k}{m}x$, thus acceleration is greatest when the spring is compressed/stretched the greatest. For an oscillation with amplitude $A$, the greatest acceleration is at $x = \pm A$

9-15

When a 30 $N$ mass is applied to a spring, it stretches 0.2 $m$.

(a) If a 5 $kg$ mass is hung from the spring and remains at rest, how much is the string stretched from its original length?

Knowing that the spring stretches 0.2 $m$ with 30 $N$ applied to it, we can solve for the spring constant.

$$F = kx \rightarrow 30 N = k(0.2 m) \rightarrow k = 150 \text{Nm}^{-1} \tag{1}$$

Now that we have $k$, we can solve for the distance any force will stretch the spring. For a 5 $kg$ mass hanging on the spring, the resulting stretching is:

$$mg = kx \rightarrow (5 \text{kg})(9.8 \text{ms}^{-2}) = (150 \text{Nm}^{-1})(x) \rightarrow x = .326 \text{m} \tag{2}$$

(b) What is the period of oscillation of the mass and the spring?

The period is defined as:

$$T = \frac{1}{f} = \left(\frac{1}{2\pi}\sqrt{\frac{k}{m}}\right)^{-1} = (2\pi)\sqrt{\frac{m}{k}}$$

$$T = (2\pi)\sqrt{\frac{5 \text{kg}}{150 \text{Nm}^{-1}}} = 1.15 \text{ s} \tag{3}$$

9-41

A 50 $kg$ boy rides on a Pogo stick, a pole with a spring on its bottom. He jumps into the air 0.3 $m$ and lands on the ground, compressing the spring .05 $m$.

(a) How much energy is stored in the spring?

By the conservation of energy, all of the gravitational potential energy at the peak of the jump must be converted to spring potential energy when the boy hits the ground.

$$U_{spring} = GPE = mgh = (50 \text{kg})(9.8 \text{ms}^{-2})(0.3 \text{m}) = 147 \text{ J} \tag{4}$$
(b) What is the spring constant?

\[
U_{spring} = \frac{1}{2} k x^2 = 147 \text{ J}
\]

\[
\frac{1}{2} k (0.05 \text{ m})^2 = 147 \text{ J}
\]

\[
k = \frac{(2)(147 \text{ J})}{(0.05 \text{ m})^2} = 117,600 \text{ Nm}^{-1}
\]

(c) What is the characteristic frequency of oscillation?

\[
f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{117,600 \text{ Nm}^{-1}}{50 \text{ kg}}} = 7.72 \text{ Hz}
\]

9-42

The otolith in a fish has a mass of .022 g = 2.2 \times 10^{-5} \text{ kg}, and the effective spring constant is 3 Nm^{-1}.

(a) What is the characteristic frequency of the otolith?

We can calculate the frequency using the same procedure as before:

\[
f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{3 \text{ Nm}^{-1}}{2.2 \times 10^{-5} \text{ kg}}} = 58.77 \text{ Hz}
\]

This oscillation is a damped. This means that over time the amplitude will decrease, as will the frequency. Therefore this frequency will only be correct very shortly after the onset of oscillation.

(b) Is the frequency consistent with the idea that the otolith should respond rapidly to changes in orientation?

The period corresponding to this frequency is \( T = 1/f = .017 \) s. This is a very short amount of time, which agrees with the idea that the otolith responds rapidly to changes in orientation.

9-55

The human leg can be approximated by a cylinder.

(a) Estimate the characteristic frequency of your legs, when swung from the hip with the knee locked.

In this problem we are treating the human leg as a physical pendulum. The frequency associated with a physical pendulum is:

\[
f = \frac{1}{2\pi} \sqrt{\frac{mgd}{I}}
\]

Where \( I \) is the moment of inertia for the object that is oscillating. If we approximate walking motion as a rod (the leg) rotating about its end, we can then directly apply the formula above. The moment of inertia of a rod or cylinder rotating about its end it \( I = \frac{1}{4} ml^2 \). \( d \) is the distance from the rotation axis to the center of gravity of the object. For
simplicity let us take $d = l/2$. Plugging in to (8)

$$f = \frac{1}{2\pi} \sqrt{\frac{mgd}{I}}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{mg l^2}{1/3 ml^2}}$$

$$f = \frac{1}{2\pi} \sqrt{\frac{3g}{2l}}$$

(9)

For the length of a leg, let's use $l = 1\ m$. The frequency can now be calculated:

$$f = \frac{1}{2\pi} \sqrt{\frac{3(9.8 \text{ m/s}^2)}{2(1.25 \text{ m})}} = 0.61\ Hz$$

(10)

(b) If normal walking were performed with the legs swinging at their natural frequency, how far could you walk in 1 hour?

The period corresponding to the frequency calculated above is $T = 1/0.61 = 1.64\ s$. This is how long it takes to complete one stride. In order to calculate the distance travelled in an hour we need the distance covered in a single stride. Walking briskly, each stride is $\approx 2\ m$. Therefore:

$$\left(\frac{1\ \text{stride}}{1.64\ s}\right)\left(\frac{2\ m}{1\ \text{stride}}\right)\left(\frac{3600\ s}{1\ \text{hr}}\right) = 4400\ m = 4.4\ km$$

(11)

9-69

The simple harmonic oscillator provides a good model for small oscillations about equilibrium in many systems, including molecules. In the $H_2$ molecule, the two hydrogen atoms can oscillate toward and away from each other, so that the center of mass remains stationary. Each moves as though connected to a spring with constant $1130\ \text{Nm}^{-1}$.

(a) Find the frequency of oscillation.

Using $m_H = 1.67 \times 10^{-27}\ \text{kg}$:

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{1130\ \text{N/m}^{-1}}{1.67 \times 10^{-27}\ \text{kg}}} = 1.3 \times 10^{14}\ Hz$$

(12)

(b) If the vibrational energy of the molecule is $1.23\ \text{eV}$, what is the amplitude of the oscillations for each atom?

Each Hydrogen atom has half of the energy of the molecule. When the oscillations are at their maximum amplitude, all of the energy of the atom is stored in the 'spring':

$$E_{\text{atom}} = \frac{1}{2} kA^2$$

$$E_{\text{atom}} = P E_{\text{spring}} = \frac{1}{2} kA^2$$

$$0.615\ \text{eV} = 9.85 \times 10^{-20}\ \text{J} = \frac{1}{2} (1130\ \text{N/m}^{-1})A^2$$

$$A = \sqrt{\frac{2(9.85 \times 10^{-20}\ \text{J})}{1130\ \text{N/m}^{-1}}} = 1.32 \times 10^{-11}\ \text{m}$$

(13)

Notice that we had to convert to Joules so that the units would end up correct
(c) Find the maximum velocity of the atoms relative to the center of mass.

Given an equation for SHM \( x(t) = A \cos(\omega t) \), the velocity as a function of time is the time derivative of the position. \( v(t) = -\omega A \sin(\omega t) \). The maximum value that \( v(t) \) can reach is \( v_{\text{max}} = \omega A \).

\[
v_{\text{max}} = \omega A = \sqrt{\frac{1130 \text{ Nm}^{-1}}{1.67 \times 10^{-27} \text{ kg}}} (1.32 \times 10^{-11} \text{ m}) = 10861 \text{ m s}^{-1}
\] (14)

9-73

Show that \( x(t) = A \cos(\omega t + \phi) \) is a solution of the SHM equation \( a = -\frac{k}{m} x \).

\( a(t) = \frac{d^2 x(t)}{dt^2} \), so to test our function we must take two time derivatives:

\[
\frac{d}{dt} (A \cos(\omega t + \phi)) = -\omega A \sin(\omega t + \phi)
\]

\[
\frac{d^2}{dt^2} (A \cos(\omega t + \phi)) = \frac{d}{dt}(-\omega A \sin(\omega t + \phi)) = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x(t)
\] (15)

By definition, \( \omega = \sqrt{\frac{k}{m}} \rightarrow \omega^2 = \frac{k}{m} \)

\[
a(t) = -\omega^2 x(t) = -\frac{k}{m} x(t)
\] (16)

Our choice of \( x(t) = A \cos(\omega t + \phi) \) as a solution to the SHM equation is correct.

(b) What is the significance of \( \phi \).

\( \phi \) is a constant, known as the phase. It describes the starting point of the oscillation. The equation \( x(t) = A \cos(\omega t) \) only describes an oscillation has \( x(0) = A \). The introduction of \( \phi \) allows us to describe an oscillation with any starting position.

9-82

Since various parts of the body have characteristic frequencies of vibration, we can think of these parts as being connected by springs. The spring is formed by the flexible connections of these parts of the body. In Ch. 8 we learned that the spring constant is proportional to the cross sectional area divided by the length of the spring material.

(a) Using scaling hypothesis of 8.6 \( l \propto r^{2/3} \), show that the spring constant \( k \propto m^{1/2} \).

From Ch. 8 we know that \( k = \frac{EA}{l} \). Using \( A \propto r^2 \), and \( l \propto r^{2/3} \rightarrow r \propto l^{3/2} \) we can see how \( k \) scales:

\[
k \propto \frac{r^2}{l} \propto (l^{3/2})^2 l \propto l^2
\] (17)

The scaling law in 8.6 lead to the conclusion that \( m \propto l^4 \rightarrow l \propto m^{1/4} \). Therefore:

\[
k \propto l^2 \propto m^{1/2}
\] (18)

(b) Show that the characteristic frequency should scale as \( f \propto m^{-1/4} \).

\[
f \propto \sqrt{km} \propto \sqrt{\frac{m^{1/2}}{m}} \propto \sqrt{m^{-1/2}} \propto m^{-1/4}
\] (19)
9-83

The abdomen and thorax of a 60 kg human has a resonance at about 3 Hz.

(a) Using the result of problem 9-82, what would you expect the corresponding characteristic frequency to be in a 20 g = 2^{-2} kg mouse?

From the previous problem, we know that \( f \propto m^{-1/4} \). We can write this relationship as \( k = cm^{-1/4} \), where \( c \) is a constant. We can solve for \( c \) using the numerical values for the resonant frequency of a human abdomen:

\[
f = cm^{-1/4} \rightarrow c = \frac{f}{m^{-1/4}} = \frac{3 \text{ Hz}}{(60 \text{ kg})^{-1/4}} = 8.35 \text{ kg}^{1/4} s
\]  

Applying the relationship to the mouse, with the correct value of \( c \) gives us the guess for the resonant frequency of a mouse abdomen:

\[
f_{\text{mouse}} = cm_{\text{mouse}}^{-1/4} = (8.35 \text{ kg}^{1/4} s)(2.0 \times 10^{-2} \text{ kg})^{-1/4} = 22 \text{ Hz}
\]

(b) Experimentally the frequency in mice is between 18 and 25 Hz. How does this compare to the result in part (a)?

The calculated value of 22 Hz is within the measured range. The scaling law for resonant frequencies of body parts is correct in this experiment.