Today: Applications

1. Which linear maps are isomorphisms?
2. Which polynomial best fits a given curve?
3. Which polynomial best fits some given data?
4. $\mathbb{H}$ and multilinear maps on $\mathbb{R}^4$?
5. Solve $\frac{d^3 f}{dx^3} - \frac{d^2 f}{dx^2} + \frac{df}{dx} - f = 0$.
6. What eigenvalues can a self-adjoint operator have?
7. Prove Taylor's theorem.
8. What are the metric preserving functions in $\mathbb{R}^n$?
9. What are the finite symmetries of the plane?

But first...

$$\left.\begin{array}{l}\text{Tensors}\\ \text{Wedge products}\\ \text{Inner product spaces}\\ \text{Operators}\end{array}\right\}\ \text{Very briefly}$$

+ an ORACLE

We already have lots of ways to create vector spaces:

$S \longrightarrow V$	free construction
$V \oplus W$	categorical sum / product
$\mathrm{Ker}\, \varphi$	Kernel of any morphism
$\mathrm{Im}\, \varphi$	Image of any morphism
V/W	quotient by a subspace
V^*	dual
$\mathrm{Span}(S)$	span of a subset
$\mathrm{Mor}(V, W)$	linear maps
$\mathrm{Mor}(V, V)$	"operators"

$$\left.\begin{array}{ll} V \otimes W & \text{tensor products} \\ V \wedge V & \text{wedge products} \end{array}\right\} \text{today.}$$

Tensor products

$$V \times W \longrightarrow \text{Free}(V \times W) \longrightarrow \text{Free}(V \times W)/M$$

$$(v, w) \qquad a_1(v_1, w_1) + a_2(v_2, w_2) + \dots + a_k(v_k, w_k)$$

$$a_1(v_1, w_1) + a_2(v_2, w_2) + \dots + a_k(v_k, w_k) + M$$

$$\equiv \; a_1 v_1 \otimes w_1 + a_2 v_2 \otimes w_2 + \dots + a_k v_k \otimes w_k$$

M forces bilinearity e.g. $(v + v') \otimes w = v \otimes w + v' \otimes w$

If $\epsilon_1, \dots, \epsilon_n$ is a basis of V and
$\eta_1, \dots, \eta_m$ is a basis of W, then

$$\{ \epsilon_i \otimes \eta_j : i \in 1, \dots, n, \; j \in 1, \dots m \}$$

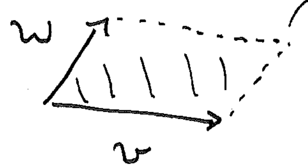
is a basis of $V \otimes W$.

<u>Example</u>: $\displaystyle\sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}(x)\, dx_i \otimes dx_j$

Riemannian metric on a manifold.

Wedge products are nicer

Idea:



signed area (v, w)

bilinear and antisymmetric

$\text{area}(v, w) = -\text{area}(w, v)$

$$V \times V \longrightarrow \mathcal{F}ree(V \times V) \longrightarrow \mathcal{F}ree(V \times V)/A$$

(v, w) $\qquad a_1(v_1, w_1) + a_2(v_2, w_2) + \dots + a_k(v_k, w_k)$

$$a_1(v_1, w_1) + a_2(v_2, w_2) + \dots + a_k(v_k, w_k) + A$$

$$\equiv a_1 v_1 \wedge w_1 + a_2 v_2 \wedge w_2 + \dots + a_k v_k \wedge w_k$$

A forces $\wedge$ to be bilinear <u>and</u> antisymmetric.

$$v \wedge w = -w \wedge v \Rightarrow v \wedge v = 0.$$

<u>Example</u>: $dx \wedge dy \wedge dz \in \mathbb{R}^* \wedge \mathbb{R}^* \wedge \mathbb{R}^*$

is the volume element in $\mathbb{R}^3$.

Compare for V n-dimensional

$$\dim(V \otimes V \otimes \cdots \otimes V) = n^n$$

$$\dim(V \wedge V \wedge \cdots \wedge V) = 1 \quad !$$

Why?

Let $\epsilon_1, \dots, \epsilon_n$ be a basis of V, $w \in \Lambda^n V$

$$w = \left(\sum_{i=1}^{n}{}' a_i \epsilon_i\right) \wedge \left(\sum_{j=1}^{n}{}' b_j \epsilon_j\right) \wedge \cdots \wedge \left(\sum_{k=1}^{n}{}' z_k \epsilon_k\right)$$

$$= \text{Const.} \times \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n$$

$\Rightarrow \underbrace{V \wedge V \wedge \cdots \wedge V}_{n}$ is one dimensional.

Let $\varphi(v_1 \wedge v_2 \wedge \cdots \wedge v_n) \equiv \varphi(v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n)$

$= K_\varphi \, v_1 \wedge v_2 \wedge \cdots \wedge v_n$ because $\Lambda^n V$ is 1-dimensional.

<u>def</u>: K_φ is called the <u>determinant</u> of φ.

Note that K_φ only depends on φ and

$\det(\psi \circ \varphi) = \det(\psi) \cdot \det(\varphi)$.

Inner product space ($F = \mathbb{R}$ or $\mathbb{C}$)

$\langle , \rangle : V \times V \to F$

$\langle v + av', w \rangle = \langle v, w \rangle + a \langle v', w \rangle$

$\langle v, w \rangle = \langle w, v \rangle^*$

$\langle v, v \rangle = 0$ iff $v = 0$

examples: $\langle v, w \rangle \equiv v_x w_x + v_y w_y + v_z w_z$ in $\mathbb{R}^3$,

$\langle v, w \rangle \equiv v_x w_x^* + v_y w_y^* + v_z w_z^*$ in $\mathbb{C}^3$,

$\langle f, g \rangle \equiv \int_a^b f(x) g(x)\, dx$ in $C[a, b]$.

Normed space

$\| \; \| : V \to \mathbb{R}^{\geq 0}$

$\| av \| = |a| \, \|v\|$

$\| v + w \| \leq \|v\| + \|w\|$

$\| v \| = 0$ iff $v = 0$

Metric space (X).

$d : X \times X \to \mathbb{R}^{\geq 0}$

$d(x, y) = d(y, x)$

$d(x, y) + d(y, z) \geq d(x, z)$

$d(x, y) = 0$ iff $x = y$

∞ dimensional spaces

Hilbert	Banach
inner product	—
norm	norm
metric	metric
complete	complete

Facts about inner product spaces

- $\|v\| \equiv \langle v, v \rangle^{1/2}$ is a norm.
- $d(v,w) \equiv \|v-w\|$ is a metric.
- $|\langle v,w\rangle| \leq \|v\|\|w\|$ Cauchy-Schwarz
- If $\langle v,w\rangle = 0$, $\|v+w\|^2 = \|v\|^2 + \|w\|^2$ Pythagoras
- If $\mathcal{U} \hookrightarrow V$, $\mathcal{U}_\perp \equiv \{v \in V : \langle v, u\rangle = 0 \text{ for all } u \in \mathcal{U}\}$ is complementary to $\mathcal{U}$ in V.
- If $\langle x, v\rangle = 0$ for all $x \in V$, $v = 0$ ($\langle , \rangle$ is <u>nondegenerate</u>)
- Orthonormal bases exist Gram-Schmidt
- $v \mapsto \langle \circ, v\rangle$ is a 1-1 morphism from V to V^*. It is an isomorphism if V is finite-dimensional.
- Adjoints exist. For any L in a finite dimensional space, there is an L^+ s.t. $\langle Lv, w\rangle = \langle u, L^+ w\rangle$ for all $v, w \in V$
  $(L+M)^+ = L^+ + M^+$, $L^{++} = L$, $(LM)^+ = M^+ L^+$

<u>def</u>: If $L^+ = L$, L is <u>self adjoint</u>.

<u>def</u>: If $L^+ L = L L^+$, L is <u>normal</u>.

Operators and Spectra

<u>def</u>: If $Lv = \lambda v$ for some nonzero v in V, then v is called an <u>eigenvector</u> of L with <u>eigenvalue</u> $\lambda \in F$.

<u>Theorem</u>: Every operator on a complex n-dimensional vector space $(n > 0)$ has an eigenvalue.

<u>Proof</u>: Choose $v \neq 0$. Then

$$v,\ Lv,\ L^2 v,\ \dots,\ L^n v$$

must be dependent, so $(a_0 + a_1 L + a_2 L^2 + \dots + a_n L^n) v = 0$ for some not-all-zero $a_0, a_1, \dots, a_n$. $\Rightarrow$

$$c(L - \lambda_1)(L - \lambda_2) \cdots (L - \lambda_n)(v) = 0 \qquad c \neq 0$$

$\Rightarrow$ L has an eigenvalue.

<u>Theorem</u>: Eigenvectors with different eigenvalues are independent.

<u>Proof</u>: hw

The two main spectral theorems

Let L be an operator on a finite-dimensional vector space V over a field F.

$F = \mathbb{C}$

<u>Thm</u>: L has an orthonormal basis of eigenvectors iff $L^{+}L = LL^{+}$.

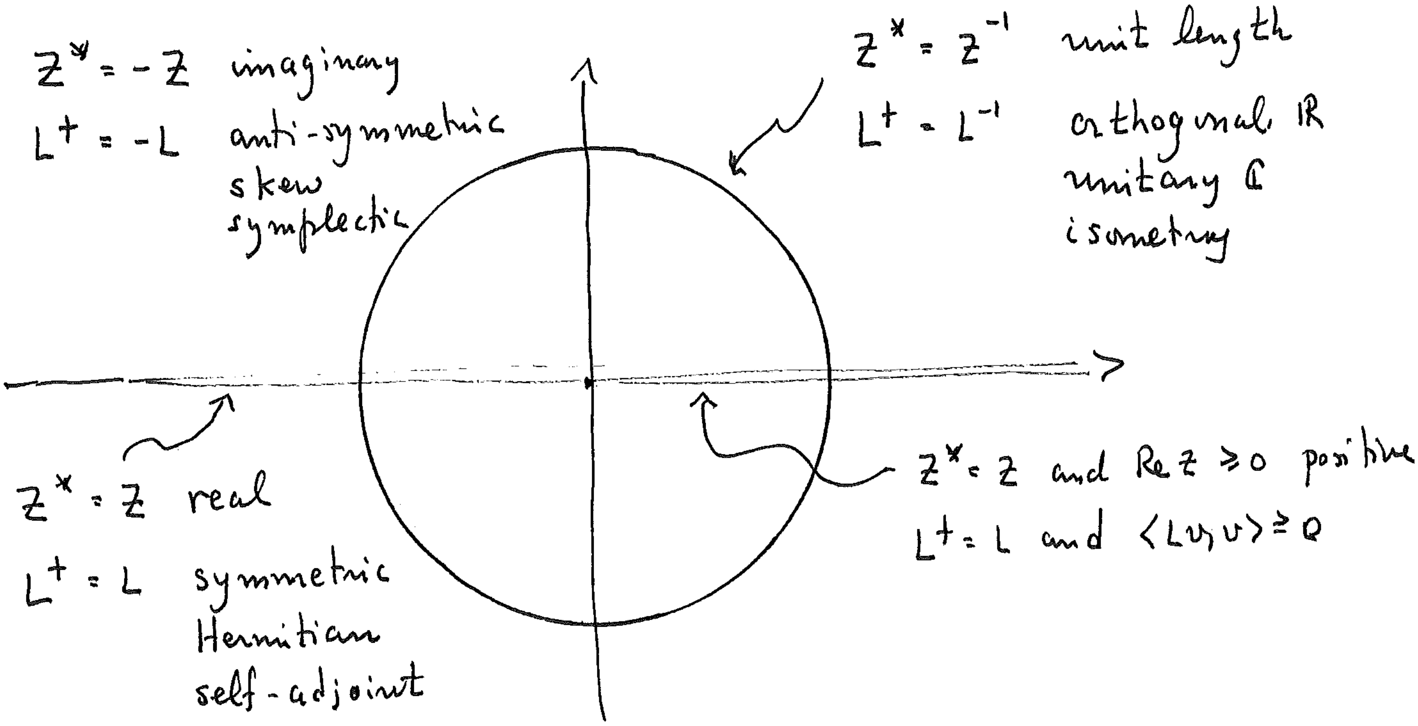
$F = \mathbb{R}$

<u>Thm</u>: L has an orthonormal basis of eigenvectors iff $L = L^{+}$.

It's all about the adjoint.

- Proofs in a separate handout.

An ORACLE



<u>Fact</u>: If z is positive, z has a positive square root.

<u>Prediction</u>: If L is positive, L has a positive square root

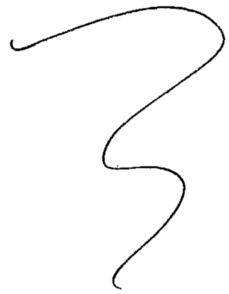
<u>Fact</u>: $z = e^{i\theta}(z^* z)^{1/2}$ for some θ.

<u>Prediction</u>: $L = S(L^+ L)^{1/2}$ for some isometry S.

<u>Fact</u>: If z is imaginary, e^z has unit length.

<u>Prediction</u>: If L is skew, e^L is orthogonal.

APPLICATIONS



1. Which operators are isomorphisms?

Consider only an operator L on an n-dimensional vector space V.

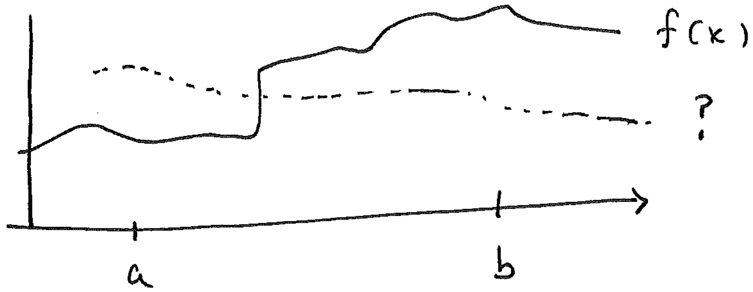
<u>Thm</u>: L is an isomorphism iff $\det(L) \neq 0$.

<u>Proof</u>: If L is an isomorphism, then
$\det(LL^{-1}) = \det(L)\det(L^{-1}) = 1$, $\Rightarrow \det(L) \neq 0$.
Conversely, suppose that L is not an isomorphism. Let nonzero $x \in \mathrm{Ker}(L)$.
Let $x, v_2, v_3, \dots, v_n$ be a basis of V.
$L(x \wedge v_2 \wedge v_3 \wedge \dots \wedge v_n) = \det(L)\,(x \wedge v_2 \wedge v_3 \wedge \dots \wedge v_n) = 0$.
$\Rightarrow \det(L) = 0$.

<u>Bonus</u>: λ is an eigenvalue of L iff $\det(L-\lambda) = 0$.

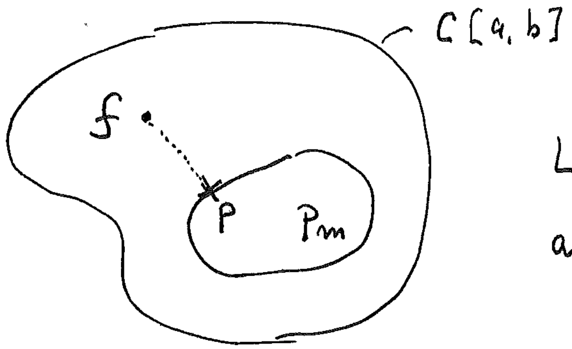
<u>Proof</u>: hw

2. What is the best polynomial approximation to a given curve?



$C[a,b]$ is an inner product space with $\langle f, g\rangle \equiv \int_a^b f(x)g(x)\,dx$.

Let $P_m \equiv$ Polynomials with degree $\leq m$



Find $p \in P_m$ closest to f.

Let $\epsilon_0, \epsilon_1, \dots, \epsilon_m$ be an orthonormal basis of P_m.

ϵ_i are Legendre Polynomials.

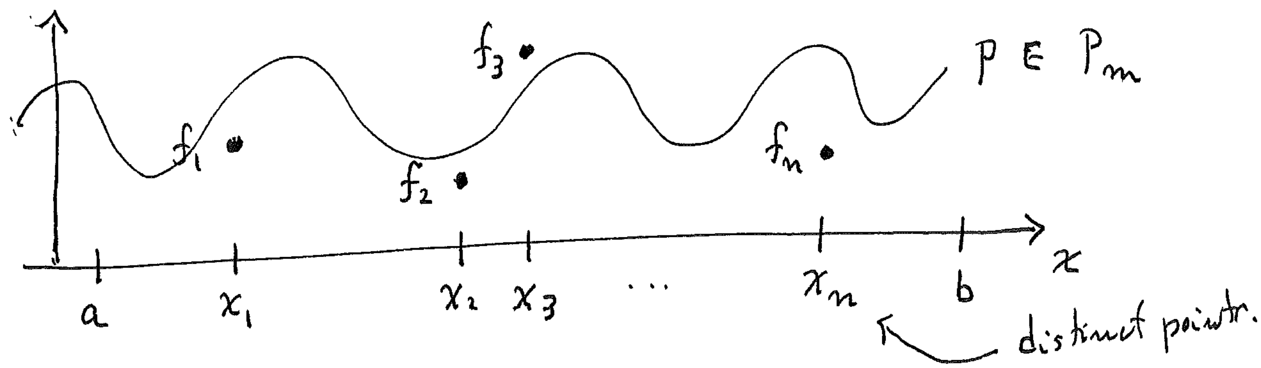
$$f = \underbrace{f - \sum_{i=0}^{m} \epsilon_i \langle \epsilon_i, f\rangle}_{\in P_m^{\perp}} + \underbrace{\sum_{i=0}^{m} \epsilon_i \langle \epsilon_i, f\rangle}_{\equiv f_p \in P_m}$$

$$\| f - p \|^2 = \| f - f_p + f_p - p \|^2 = \| f - f_p \|^2 + \| f_p - p \|^2$$

$\Rightarrow$ $p = f_p$ is the unique minimum.

$\Rightarrow$ $p = \sum_{i=0}^{m} \epsilon_i \langle \epsilon_i, f \rangle$ is the answer.

3. What is the best polynomial fit to given data?



Let $f E g$ if f and g agree on $x_1, x_2, \ldots, x_n$.

$$\left.\begin{aligned} [f] + [g] &\equiv [f+g] \\ a \cdot [f] &\equiv [af] \end{aligned}\right\} \quad C[a,b]/E$$

is still a vector space with inner product

$$\langle [f], [g] \rangle \equiv \sum_{i=1}^{n} f(x_i)\, g(x_i)$$

<u>Claim</u>: $[1], [x], [x^2], \ldots, [x^m]$ is a basis of P_m/E.

<u>Proof</u>: Suppose $a_0[1] + a_1[x] + \ldots + a_m[x^m] = 0 \in P_m/E$.

$\Rightarrow [a_0 1 + a_1 x + a_2 x^2 + \ldots + a_m x^m] = [x \mapsto 0]$

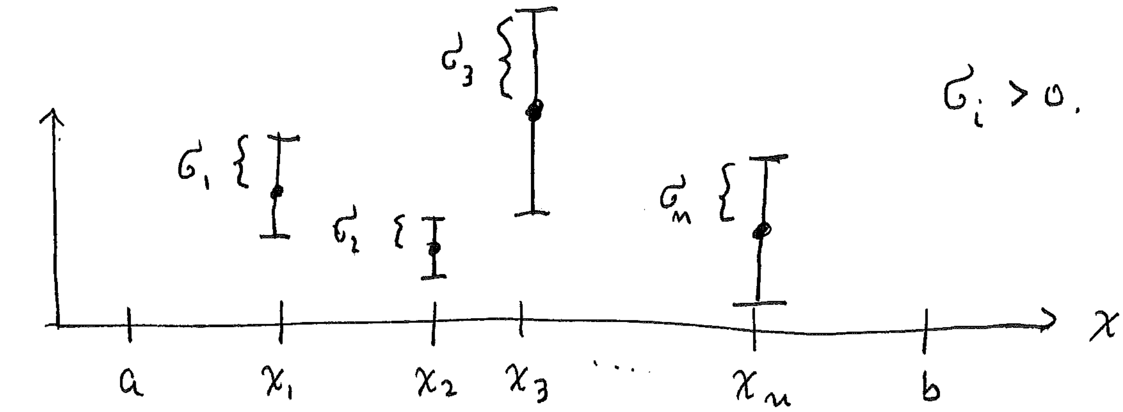
$\Rightarrow a_0 = a_1 = \cdots = a_m = 0$, $\boxed{\text{provided } m < n}$.

Let $[\epsilon_0], [\epsilon_1], \ldots, [\epsilon_m]$ be a basis of P_m/E that is orthonormal.

$\Rightarrow p = \sum_{i=0}^{m} [\epsilon_i] \langle [\epsilon_i], [f] \rangle$ is the answer again!

$$= \left[\sum_{i=0}^{m} \sum_{j=1}^{n} \epsilon_i\, \epsilon_i(x_j)\, f_j \right].$$

What about "error bars"?



Just redefine $\langle [f], [g] \rangle \equiv \sum_{i=1}^{n} f(x_i)\, g(x_i) / \sigma_i^2.$

$$\Rightarrow \quad p = \left[\sum_{i=0}^{m} \sum_{j=1}^{n} e_i \; e_i(x_j)\, f_j / \sigma_j^2 \right].$$

Note that if you apply Gram-Schmidt to the basis $1, x, x^2, \ldots, x^m$ of P_m, you get the Legendre polynomials.

4. What are the bilinear forms on $\mathbb{R}^4$?

You might have seen the quaternions defined like this:

"$i^2 = j^2 = k^2 = ijk = -1$."   Huh?

$\mathbb{H} = \mathbb{R}^4$

$$\begin{tikzcd} \mathbb{H} \times \mathbb{H} \arrow[r] \arrow[dr, "\text{bilinear multiplication}"'] & \mathbb{H} \otimes \mathbb{H} \arrow[d, "\varphi"] \\ & \mathbb{H} \end{tikzcd}$$

extends to a unique map

$$\begin{tikzcd} \{1,i,j,k\} \times \{1,i,j,k\} \arrow[r] \arrow[dr] & \mathbb{H} \otimes \mathbb{H} \arrow[d, "\varphi"] \\ & \mathbb{H} \end{tikzcd}$$

bilinear maps from $\mathbb{H} \times \mathbb{H}$ to $\mathbb{H}$ $\cong$

linear maps from $\mathbb{H} \otimes \mathbb{H}$ to $\mathbb{H}$ $\cong$

functions from $\{1,i,j,k\} \times \{1,i,j,k\}$ to $\mathbb{H}$

$\Rightarrow$ 64 dimensional space of bilinear forms

$\Rightarrow$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

defines quaternion multiplication.

5. Solve $\frac{d^3 f}{dx^3} - \frac{d^2 f}{dx^2} + \frac{df}{dx} - f = 0$.

Actually, we might as well solve $Df = 0$ where

$$D \equiv a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots + a_1 \frac{d}{dx} + a_0$$

Letting V be the vector space of smooth functions from $[a,b]$ to $\mathbb{R}$, $L \equiv \frac{d}{dx}$ is a linear operator.

$$\Rightarrow \quad D = c\,(L-\lambda_1)^{m_1}(L-\lambda_2)^{m_2} \cdots (L-\lambda_k)^{m_k}$$

for polynomial roots λ_i with corresponding multiplicities m_i, $\lambda_i, c \in \mathbb{C}$. Using

<u>Fact</u>: If polynomials p and q have no common factors, then $\operatorname{Ker}(p(L)\cdot q(L)) \cong \operatorname{Ker}(p(L)) \oplus \operatorname{Ker}(q(L))$.

$$\operatorname{Ker} D = \operatorname{Ker}(L-\lambda_1)^{m_1} \oplus \operatorname{Ker}(L-\lambda_2)^{m_2} \oplus \dots \oplus \operatorname{Ker}(L-\lambda_k)^{m_k}$$

Since $\operatorname{Ker}(L-\lambda)^N = \operatorname{span}\{e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \dots, x^{N-1}e^{\lambda x}\}$, we have all solutions to $Df = 0$. They form an $m_1 + m_2 + \dots + m_k = n$ dimensional subspace of V.

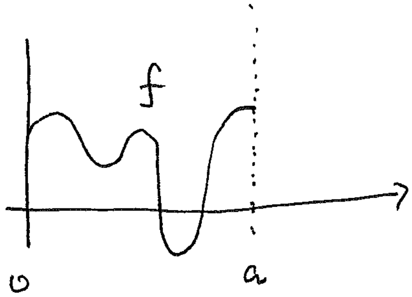
6. What eigenvalues can a self-adjoint operator have?

Suppose $Lv = \lambda v$, $v \neq 0$.

$$\langle Lv, v \rangle = \lambda \langle v, v \rangle = \langle v, Lv \rangle = \lambda^* \langle v, v \rangle$$

$\Rightarrow \lambda = \lambda^*$, λ is real.

7. Taylor's theorem



$V \equiv$ Continuous, real valued functions on $[0, a]$.

$$\langle f, g \rangle \equiv \int_0^a f(x)\, g(x)\, dx$$

$$\int_0^a \frac{d}{dx}(f \cdot g)\, dx = fg \Big|_0^a = \left\langle \frac{df}{dx}, g \right\rangle + \left\langle f, \frac{dg}{dx} \right\rangle$$

Let $f_n \equiv \dfrac{d^n f}{dx^n}$, $\quad P_n \equiv \dfrac{(a-x)^n}{n!}$ so that $\dfrac{dP_n}{dx} = -P_{n-1}$

$$\Rightarrow \langle f_{n+1}, P_n \rangle = f_n P_n \Big|_0^a + \langle f_n, P_{n-1} \rangle \qquad \text{for } n \geq 1$$

$$\Rightarrow \langle f_{n+1}, P_n \rangle = f_n P_n \Big|_0^a + f_{n-1} P_{n-1} \Big|_0^a + \dots + f_1 P_1 \Big|_0^a + \underbrace{\langle f_1, P_0 \rangle}_{f(a) - f(0)}$$

$$\Rightarrow f(a) = f(0) + a f_1(0) + \frac{a^2}{2} f_2(0) + \dots + \frac{a^n}{n!} f_n(0)$$

$$+ \langle f_{n+1}, P_n \rangle.$$

Q.E.D.

8. What are the metric preserving functions on $\mathbb{R}^n$?

$d(f(x), f(y)) = d(x,y)$ for all $x, y \in \mathbb{R}^n$

Call such functions "rigid". Note:

- Translations are rigid (note: not linear!)
- If f and g are rigid, so is $g \circ f$.
- Rigid functions are 1-1.
- Any rigid motion f satisfies $\Psi = t_{-f(0)} \circ f$ where rigid Ψ fixes the origin.
- Ψ is norm preserving
  [because $\|\Psi(x)\| = \|\Psi(x) - \Psi(0)\| = d(\Psi(x), \Psi(0)) = \|x\|$].
- Ψ is inner product preserving
  [because $d(\Psi(x), \Psi(y))^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle \Psi(x), \Psi(y)\rangle$]
- Ψ is linear
  [because $\|\Psi(x+ay) - \Psi(x) - a\Psi(y)\|^2 = \|x+ay\|^2 + \|x\|^2$
  $+ a^2\|y\|^2 - 2\langle x+ay, x\rangle - 2\langle x+ay, y\rangle + 2\langle x, ay\rangle = 0$].
- $\Rightarrow \Psi \in \mathcal{O}(n) \Rightarrow f$ is onto $\Rightarrow$ rigid motions are a group.
- Because $\mathcal{O}(n) \xrightarrow{\det} \{1, -1\}$ is a morphism,

$$f = t \circ \psi \circ \delta^k \quad \text{where} \quad \begin{array}{l} k \in \{0,1\} \\ \delta \in SO(n)^- \text{ fixed} \\ \psi \in SO(n) \\ t \in \mathbb{R}^n \text{ translation.} \end{array}$$

9. What are the symmetries of the plane?

Note that translations are special in that they commute with the average of points in $\mathbb{R}^2$:

$$t_\Delta\left(\frac{x_1+x_2+\dots+x_n}{n}\right) = \frac{t_\Delta x_1 + t_\Delta x_2 + \dots + t_\Delta x_n}{n}$$

so if $x_1, x_2, \dots, x_n$ is any orbit of a finite subgroup G of rigid motions in $\mathbb{R}^2$, then

$$t_\Delta \circ \Psi\left(\frac{x_1+x_2+\dots+x_n}{n}\right) = \frac{x_1+x_2+\dots+x_n}{n} \equiv p$$

is actually a fixed point. Thus, G is isomorphic to a conjugate subgroup $t_p \circ G \circ t_p^{-1} \equiv G'$ which fix the origin. Choosing $\sigma \in SO(2)^-$,

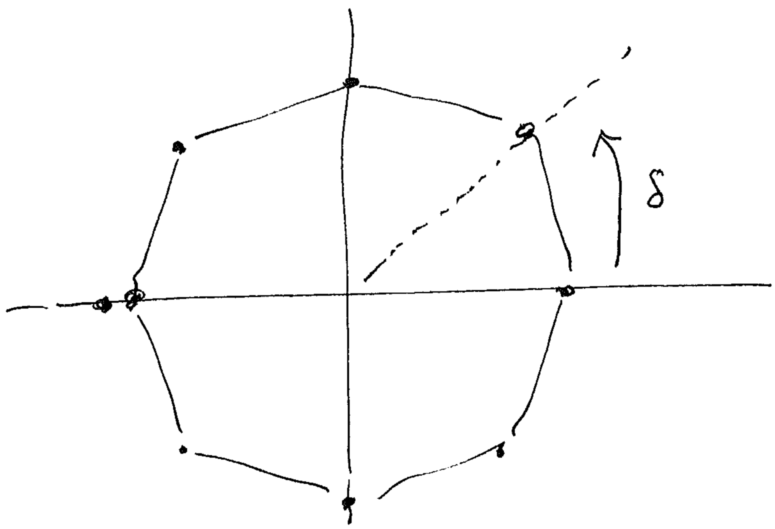
$$G' = \{ \psi_i \sigma^{K_i} : i = 1, \dots, n \} \quad \psi_i \in SO(2).$$

Let δ be the smallest positive rotation angle in $\psi_1, \psi_2, \dots, \psi_n$. $\Rightarrow$ All of $\psi_1, \dots, \psi_n$ are power of a rotation by δ. $\Rightarrow$

$$G' = C_n \rtimes \{1, \sigma\} \quad \text{where } C_n \text{ is the "cyclic group".}$$

G' is also known as $D_n \equiv C_n \rtimes \{1, \sigma\}$ the "dihedral" group.

$\mathbb{R}^2$



$\sigma : x \mapsto -x$

$D_8 \cong C_8 \rtimes \{1, \sigma\}$

<u>Bonus</u>: In $\mathbb{R}^3$, D_n are still finite subgroups of the rigid motions, however there are also additional exceptional subgroups:

T : The <u>tetrahedral group</u> of 12 rotations mapping a tetrahedron to itself.

O : The <u>octahedral group</u> of 24 rotations mapping a cube to itself

I : The <u>icosahedral group</u> of 60 rotations mapping an icosahedron to itself