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Vector Spaces

We've had:

Groups

- associative
- has identities
- has inverses

$\mathbb{Z}$, $\mathbb{R}^{\neq 0}$, $\mathbb{R}^{>0}$, $GL_n(\mathbb{C})$, $\mathcal{O}(n)$, $SU(2)$, $\mathbb{H}^{*}$, $Aut(X)$ in any category

Monoids

- associative
- has identities

$\mathbb{N}$, matrices, $End(X)$ in any category

Rings

- Abelian group with $+$
- monoid with $\cdot$
- distributes both ways

$$a \cdot (b+c) = a \cdot b + a \cdot c$$
$$(b+c) \cdot a = b \cdot a + c \cdot a$$

$\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $M_n(\mathbb{R})$, real functions on $[0,1]$, $\mathbb{Z}[x]$, $\mathbb{C}[x]$

Fields

- A commutative ring where every non zero element has a multiplicative inverse.

$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$

<u>def</u>: A <u>vector space</u> over a field F is an abelian group V with a bi-additive $\cdot : F \times V \to V$ satisfying $a \cdot (b \cdot v) = (ab) \cdot v$ and $1 \cdot v = v$.

<u>Examples</u>: $\mathbb{R}^n$, $\mathbb{C}^n$, $(\mathbb{Z}_p)^n$, infinite sequences of reals, real valued functions on a set S with pointwise addition and real scalar multiplication.

Let's use some of these to illustrate the categorical "free construction".

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & X \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

The diagram commutes for a unique morphism

Given any f

Category of sets | Some new category

(α, X) is a free <new thing> if, for any f, Y, the diagram commutes for a unique morphism $\bar{f}$.

hw: Use general abstract nonsense to prove that free constructions are unique in any category.

$S \equiv \{a, b, c\}$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & M \arrow[d, "\bar{f}"] \\ & N \end{tikzcd}$$

Free monoid on the set S

$abbcaca \in M$
"words with the alphabet S" with concatenation

$\alpha : s \mapsto s$

$\bar{f}(abbcaca) \overset{?}{=}$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & G \arrow[d, "\bar{f}"] \\ & H \end{tikzcd}$$

Free group on the set S

$abb^{-1}caca^{-1} \in G$

$\alpha : s \mapsto s$

$\bar{f}(abb^{-1}caca^{-1}) \overset{?}{=}$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & V \arrow[d, "\bar{f}"] \\ & W \end{tikzcd}$$

Free real vector space on S

$r_a a + r_b b + r_c c \in V$
"linear combinations of a, b, c"

$\bar{f}(r_a a + r_b b + r_c c) \overset{?}{=}$

Note that this is the same as F-valued functions which are nonzero at a finite number of points as in Geroch.

We will soon see that all vector spaces are free. First, some categorical business.

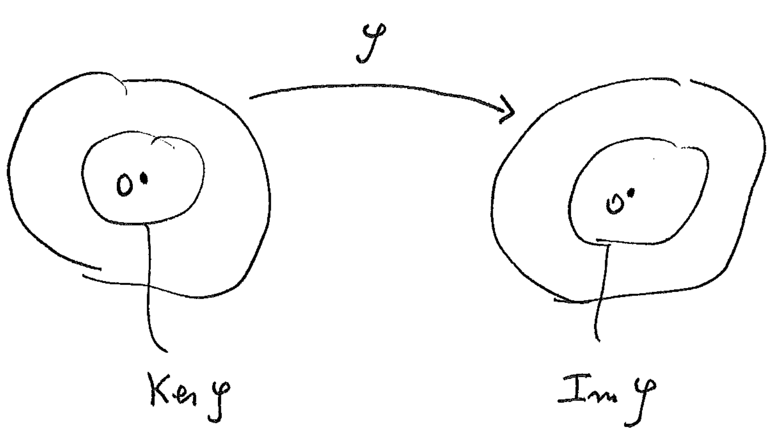
- Fix a particular field F. Let objects be vector spaces over F. Let morphisms $V \xrightarrow{\varphi} W$ be functions satisfying

$$\varphi(v + v') = \varphi(v) + \varphi(v')$$

$$\varphi(a \cdot v) = a \cdot \varphi(v)$$

Such functions are called <u>linear maps</u>. Checking that the composition of two linear maps and identity maps are linear, we have the category of vector spaces over the field F.

- Just as in the case of groups, kernels and images are subgroups



hw: prove it.

Just as in the case of groups, we have, for $V \xrightarrow{\varphi} W$

$$\begin{array}{ccccc} \text{mono} & \Longleftrightarrow & \text{1-1 linear} & \Longleftrightarrow & \operatorname{Ker}\varphi = \{0\} \\ \text{epi} & \Longleftrightarrow & \text{onto linear} & \Longleftrightarrow & \operatorname{Im}\varphi = W \\ \text{iso} & \Longleftrightarrow & \text{both} & \Longleftrightarrow & \text{both} \end{array}$$

Proof: hw

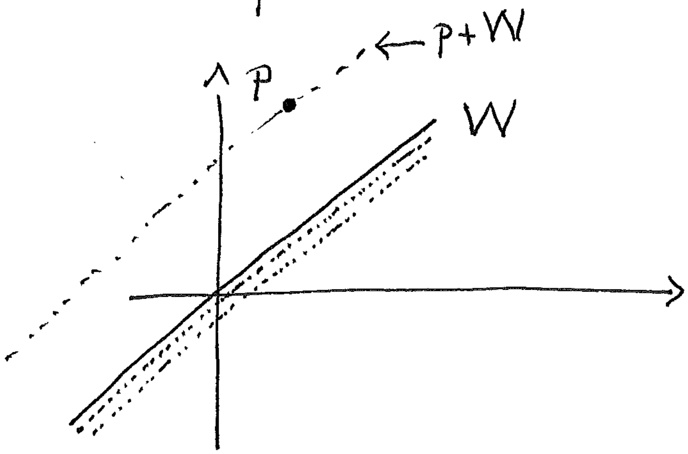
Of course, all subspaces are normal, so quotients work with any vector subspace W of V:

$$V/W \equiv \{ v + W : v \in V \}$$

$$(v+W) + (v'+W) \equiv (v+v') + W$$

$$a \cdot (v+W) \equiv (a \cdot v) + W$$

Same example as before



$V = \mathbb{R}^2$

Cosets $p + W$ cover V without overlapping

As in the case of groups, all morphisms beautifully factorize with an isomorphism inside.

$$\begin{tikzcd} V \arrow[r, "\text{epi}"] \arrow[rrr, bend left, "\varphi"] & V/\mathrm{Ker}\,\varphi \arrow[r, "\text{iso}"] & \mathrm{Im}\,\varphi \arrow[r, "\text{mono}"] & U \end{tikzcd}$$

$$v \longmapsto v + \mathrm{Ker}\,\varphi \longmapsto \varphi(v) \longmapsto \varphi(v)$$

hw: prove.

- Products and sums exist and are even nicer than with groups.

$$\begin{tikzcd}
V & V \times W \arrow[l, "\alpha"'] \arrow[r, "\beta"] & W \\
& X \arrow[u, "\gamma"] \arrow[ul, bend left, "f"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

Cartesian product with

$(v, w) + (v', w') \equiv (v+v', w+w')$

$a \cdot (v, w) \equiv (a \cdot v, a \cdot w)$

$\alpha : (v, w) \mapsto v$

$\beta : (v, w) \mapsto w$

$\gamma(x) \equiv (f(x), g(x))$ is linear.

The sum is actually the same space as the product in this category!

$$\begin{tikzcd}
V \arrow[r, "\alpha"] \arrow[dr, bend right, "f"'] & V \oplus W \arrow[d, "\gamma"] & W \arrow[l, "\beta"'] \arrow[dl, bend left, "g"] \\
& X &
\end{tikzcd}$$

$\alpha : v \mapsto (v, 0)$

$\beta : w \mapsto (0, w)$

$\gamma(v, w) \equiv f(v) + g(w)$ is linear and causes the diagram to commute.

☆ Now we are going to introduce a very powerful idea that leads to a complete classification of all vector spaces over a fixed field F.

<u>def</u>: A subset S of vector space V is <u>independent</u> if, for any distinct $s_1, s_2, \dots, s_n \in S$,

$$a_1 \cdot s_1 + a_2 \cdot s_2 + \dots + a_n s_n = 0$$

implies $a_1 = a_2 = \dots = a_n = 0$ in F.

<u>def</u>: The <u>span</u> of a subset S of vector space V is the set of finite linear combinations of elements of S.

<u>def</u>: A subset S of vector space V is a <u>basis</u> if it is independent and also spans V.

The independence / span / basis concepts parallel mono / epi / iso in the following way.

For $V \xrightarrow{\varphi} W$,

φ is mono $\Rightarrow$ φ preserves independence

φ is epi $\Rightarrow$ φ preserves span

φ is iso $\Rightarrow$ φ preserves bases

where

<u>φ preserves independence</u> means:

if $S \subset V$ is independent, so is $\varphi[S]$.

<u>φ preserves span</u> means:

if $S \subset V$ spans V, then $\varphi[S]$ spans W.

<u>φ preserves bases</u> means:

if $S \subset V$ is a basis, then so is $\varphi[S]$.

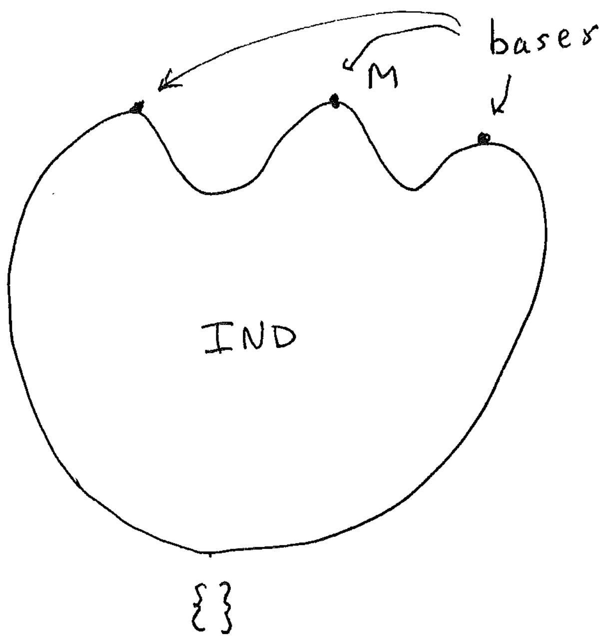
<u>Proof</u>: EZ homework.

Theorem: Every vector space has a basis.

Proof: Given a vector space V, let IND be the set of independent subsets of V, ordered by inclusion. Let I be a totally ordered subset of IND. Then

- $\cup I$ is independent
- By Zorn, IND has a maximal M
- M must also span V

Thus, every vector space has a basis.



Zorn: If every totally ordered subset of a partially ordered set has an upper bound, then the partially ordered set has a maximal element.

Cor: Every vector space is the free vector space on a basis.

Theorem: If A and B are bases of a vector space V, then $A \cong B$.

Proof: See separate note and/or Geroch

<u>def</u>: The <u>dimension</u> of a vector space V ["$\dim(V)$"] is the isomorphism class of any basis of V.

<u>Theorem</u>: $V \cong W$ iff $\dim(V) \cong \dim(W)$.

<u>Proof</u>: If $\dim(V) \overset{f}{\cong} \dim(W)$, the linear map induced by f is an isomorphism. Conversely, if $V \overset{\varphi}{\cong} W$, then φ restricted to any basis of V gives $\dim(V) \cong \dim(W)$.

☆ Thus, we have completely classified all vector spaces over a fixed field F.

- We know all of them. They are all free.
- We know exactly which ones are isomorphic

This kind of complete victory is rare in mathematics.

For finite dimensional spaces V, W, $V \xrightarrow{\varphi} W$,

$\dim(V) = \dim(\ker\varphi) + \dim(\operatorname{Im}\varphi)$.

Complementary Subspaces

def: Subspaces $\mathcal{U}$ and $\mathcal{W}$ of vector space V are *complementary* if

(a) Every v can be written as $v = u + w$ for some $u \in \mathcal{U}$, $w \in \mathcal{W}$.

(b) $u + w = 0 \Rightarrow u = w = 0$.

Thm: Complementary subspaces exist.

Proof: Geroch and/or S.Y. handout.

Factoid: If $\mathcal{U}$ and $\mathcal{W}$ are complementary in V, then $V \cong \mathcal{U} \oplus \mathcal{W}$.

Proof: Conditions (a) and (b) above are precisely that linear map $\psi : (u, w) \mapsto u + w$ has $\operatorname{Im} \psi = V$ and $\operatorname{Ker} \psi = \{(0,0)\}$.

Dual Vector Spaces

If V is a vector space,

$$V^* \equiv \{ \text{linear maps from } V \text{ to } F \}$$

with pointwise addition and scalar multiplication is called the dual vector space of V.

Example: $V = \mathbb{R}^3$

$$dx : (x, y, z) \longmapsto x$$
$$dy : (x, y, z) \longmapsto y \qquad\qquad dx, dy, dz \in V^*$$
$$dz : (x, y, z) \longmapsto z$$

This is actually a basis of $(\mathbb{R}^3)^*$.

Example: Let V be the vector space of real valued continuous functions on $[0,1]$. Fix a $p \in V$.

$$f \longmapsto \int_0^1 p(x)\, f(x)\, dx$$

is a linear map from V to $\mathbb{R}$ and therefore is an element of V^*.

$V \xrightarrow{\varphi} W$ induces a linear map $V^* \xleftarrow{\varphi^*} W^*$

defined by

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "f \circ \varphi"'] & W \arrow[d, "f"] \\ F & F \end{tikzcd}$$

You can also easily check that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & U \arrow[r, "\psi"] \arrow[d] \arrow[l, phantom, "\varphi^*" below] & W \arrow[d] \arrow[l, phantom, "\psi^*" below] \\ F & F & F \end{tikzcd}$$

$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ See Geroch for more details.

Note that we saw $\mathbb{R}^3 \cong (\mathbb{R}^3)^*$. Is this true in general?

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & V \arrow[d, "\bar{f}"] \\ & F \end{tikzcd}$$

$f \longleftrightarrow \bar{f}$ is a set isomorphism.

$V \cong \{$ functions from S to F which are zero except at a finite number of points $\}$

$V^* \cong \{$ functions from S to F in general $\}$

V^* is thus bigger as a set than V.

...For example, let

$$V \equiv \{ \text{continuous real functions on } [0,1] \}$$

$$\Psi(p) \equiv \left(f \mapsto \int_0^1 p(x) f(x)\, dx \right)$$

is a linear map from V to V^*. Ψ is 1-1, but the image of Ψ is not all of V^*. For example

$$\delta_y : f \longmapsto f(y)$$

is not in the image of Ψ. This is a "Dirac delta function" a.k.a a "distribution".

Note that this means that "bra" and "ket" notation in physics is misleading. It looks like

$$|x\rangle \longleftrightarrow \langle x|$$

is an isomorphism, but that's wrong ~~..~~ for infinite dimensional spaces.

<u>Next time:</u> Tensors and multilinear maps
Wedge product and determinant
Eigenvalues, spectral theorems
Inner product spaces
Applications