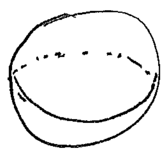
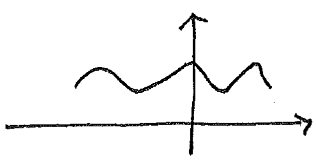
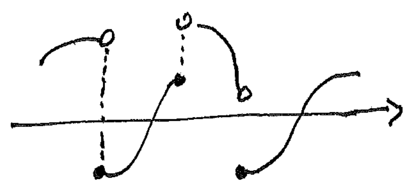
TOPOLOGY

... is about shape and continuity

 vs 

 vs 

def: A topological space (or just a space) is a set X with a distinguished collection of subsets called "open sets" which satisfy

- X and $\emptyset$ are open.
- If $\mathcal{O}_\lambda$ are open, so is $\bigcup_\lambda \mathcal{O}_\lambda$.
- If $\mathcal{O}$ and $\mathcal{O}'$ are open, so is $\mathcal{O} \cap \mathcal{O}'$.

def: A subset of X is closed if it's complement is an open set.

note: Sets can be both open and closed and may be neither open nor closed.

Examples:

- All subsets of X are open
- Only $\emptyset$ and X are open
- X is a metric space and $A \subset X$ is open iff for any $a \in A$, there is an $\epsilon > 0$ such that the open ball B_a^{ϵ} is a subset of A.
- $X = \mathbb{R}$ above is called the real line.

def: A function $X \xrightarrow{f} Y$ is continuous if the preimages of open sets in Y are open in X. ☆

You can easily verify that the composition of two continuous functions is continuous and that spaces and continuous maps form the category of topological spaces.

def: A space is Hausdorff if for any $x, y \in X$ $x \neq y$, $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$ for some open set $\mathcal{O}_x$ containing x and $\mathcal{O}_y$ containing y.

Characterizing open sets

Thm: $A \subset X$ is open iff every $a \in A$ is contained in an open subset of A.

Proof: Let $a \in A$, $a \in \mathcal{O}_a$ open subset of A. Then $A = \bigcup_a \mathcal{O}_a$ is open.

def: $\mathrm{Int}(A) \equiv$ The union of open subsets of A.

$\mathrm{Cl}(A) \equiv$ The intersection of closed supersets of A.

$\mathrm{Bdy}(A) \equiv \mathrm{Cl}(A) - \mathrm{Int}(A)$

Thm: (a) $x \in \mathrm{Int}(A)$ iff some $\mathcal{O}_x$ is a subset of A

(b) $x \in \mathrm{Cl}(A)$ iff every $\mathcal{O}_x$ intersects A.

(c) $x \in \mathrm{Bdy}(A)$ iff every $\mathcal{O}_x$ intersects both A and A^c.

The intersection of a set of topologies on a fixed set is another topology. This means that we can define the ~~th~~ topology generated by any collection of subsets A_λ of X by the intersection of all topologies which contain all of the A_λ.

For example...

<u>Thm</u>: The real line is generated by the set of open intervals in $\mathbb{R}$.

<u>Proof</u>: Let $\mathcal{T}_M$ be the standard metric topology and let $\mathcal{T}$ be the topology generated by open intervals. Since $\mathcal{T}_M$ contains the open intervals also, $\mathcal{T} \subset \mathcal{T}_M$. Since every open set in $\mathcal{T}_M$ is the union of open intervals, $\mathcal{T}_M \subset \mathcal{T}$.

Topologies induced by maps

$$X \xrightarrow{f} Y$$

set $\qquad\qquad$ topological space

$$\mathcal{T}_X \equiv \{ f^{-1}[\mathcal{O}] : \mathcal{O} \text{ is open in } Y \}$$

$$X \xrightarrow{f} Y$$

topological space $\qquad\qquad$ set

$$\mathcal{T}_Y \equiv \{ \mathcal{O} \in \mathcal{P}(Y) : f^{-1}[\mathcal{O}] \text{ is open in } X \}$$

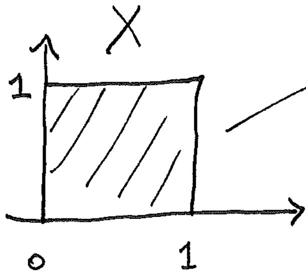
f ends up being continuous in both cases.

$$\begin{tikzcd}
X \arrow[r, "\text{epi}"] \arrow[rrr, bend left, "f"] & X/E \arrow[r, "\text{iso}"] & \mathrm{Im}\, f \arrow[r, "\text{mono}"] & Y
\end{tikzcd}$$

induced topology $\qquad\qquad$ induced topology

$\mathcal{O} \subset X/E$ is open iff $\bigcup_{x \in \mathcal{O}} [x]$ is open in X.

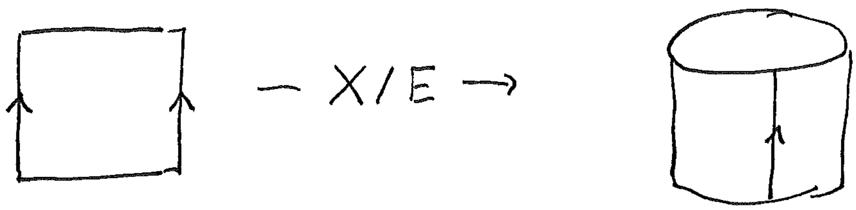
Examples



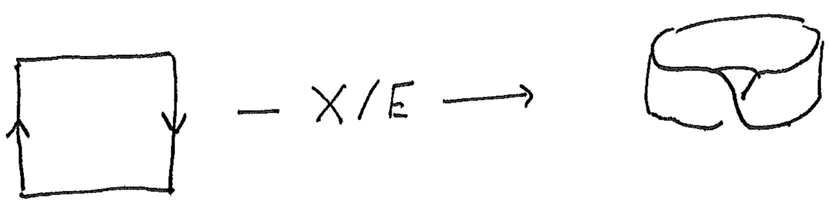
$[0,1] \times [0,1]$ with the standard topology.

Let $(0,y)\, E\, (1,y)$ for $y \in [0,1]$

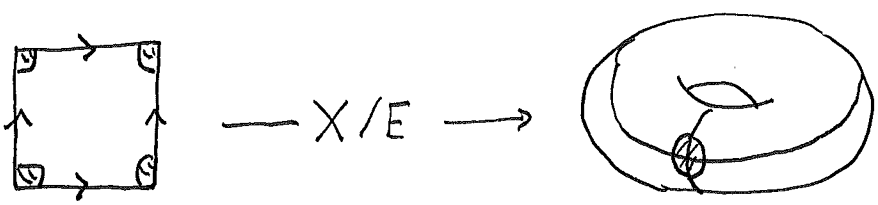
$\Leftrightarrow$ "identify opposite sides".



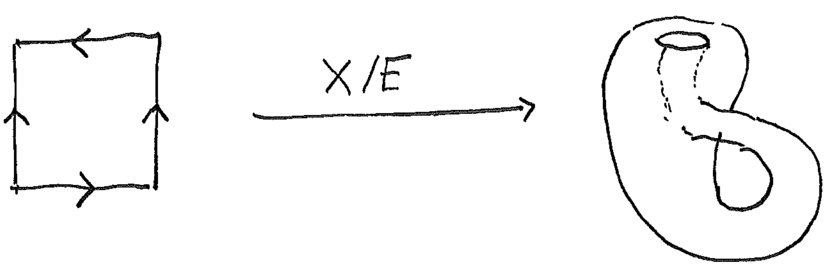
Cylinder



Möbius strip



Torus



Klein bottle

Categorical Issues

$$X \xrightarrow{\varphi} Y$$

- φ is mono iff φ is 1-1 continuous
- φ is epi iff φ is onto continuous
- φ is iso iff φ is bijective and images and preimages of open sets are open

Products and Sums exist

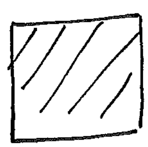
$$X \xleftarrow{\alpha} X \times Y \xrightarrow{\beta} Y \qquad \text{Cartesian product}$$

$$X \xrightarrow{\alpha} X \oplus Y \xleftarrow{\beta} Y \qquad \text{Disjoint union}$$

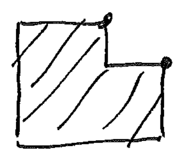
Topologies on $X \times Y$ and $X \oplus Y$ are induced by α, β.

Nets are a powerful way to characterize both open sets and continuous maps.

<u>def</u>: A partially ordered set, Δ is <u>directed</u> if, for any $\delta, \delta' \in \Delta$, there is a $\delta'' \in \Delta$ such that $\delta \leq \delta''$ and $\delta' \leq \delta''$.



directed



not directed

<u>def</u>: A <u>net</u> in a topological space X is a directed set Δ and a map $\Delta \xrightarrow{K} X$.

<u>def</u>: A net <u>converges to a limit</u> $x \in X$ if, for any $\mathcal{O}_x \ni x$, there is a $\delta \in \Delta$ s.t. $x_{\delta'} \in \mathcal{O}_x$ for all $\delta \leq \delta'$.

<u>Theorem</u>: Limits in a Hausdorff space are unique.

<u>Proof</u>: Suppose x_δ $(\delta \in \Delta)$ converges to both a and b $(a \neq b)$ in X. Let $\mathcal{O}_a \cap \mathcal{O}_b = \emptyset$ and $\delta_a, \delta_b \in \Delta$ be the guaranteed elements of Δ. Since Δ is directed, we have a $\delta \in \Delta$ s.t. $\delta_a \leq \delta$ and $\delta_b \leq \delta$. But then $x_\delta \in \mathcal{O}_a$, $x_\delta \in \mathcal{O}_b \Rightarrow \Leftarrow$. $\Rightarrow a = b$.

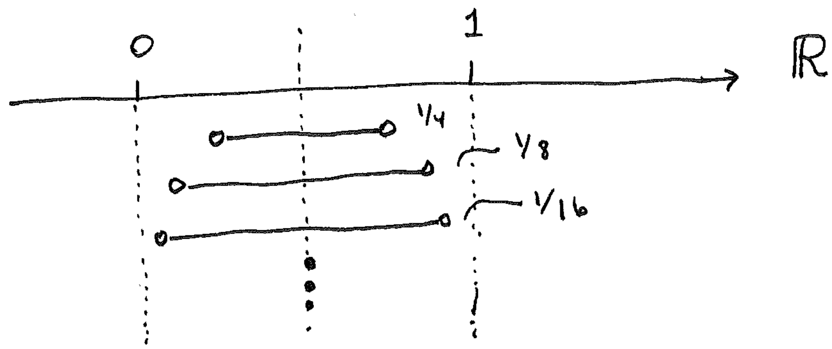
<u>Theorem</u>: Let $X \xrightarrow{\varphi} Y$ be a function.
φ is continuous iff for every net $x_\delta \to x$ in X, $(\delta \in \Delta)$
the net $\varphi(x_\delta)$ converges to $\varphi(x)$.

<u>Proof</u>: hw

Compact Spaces

<u>def</u>: A topological space is <u>compact</u> if every open cover contains a finite sub-cover.

<u>example</u>:

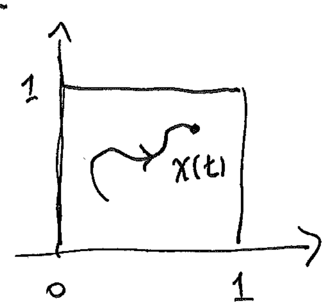


These sets cover $(0,1)$, but no finite sub-cover does the job. $\Rightarrow$ $(0,1)$ is not compact.

<u>def</u>: A net x_δ has an <u>accumulation point</u> $a \in X$ if, for every $\mathcal{O}_a \ni a$, $\delta \in \Delta$, there is a $\delta \leq \delta'$ such that $x_{\delta'} \in \mathcal{O}_a$.

<u>Theorem</u>: A space is compact iff every net has an accumulation point.

<u>Example</u>:



$X = [0,1] \times [0,1]$ is compact

$\mathbb{R}$ is a directed set

x_t is a trajectory in X

$\Rightarrow$ Every trajectory has a point of accumulation!

<u>Theorem</u>: A closed subset of a compact space is compact.

<u>Theorem</u>: A compact subset of a Hausdorff space is closed.

<u>Cor</u>: A subset of $\mathbb{R}$ is compact iff it is closed and bounded.

[called the Heine-Borel theorem].

<u>Theorem</u>: The image of a compact space is compact.

<u>examples</u>:

- $x \mapsto 1/x$ from $(0,1)$ to $\mathbb{R}$ is continuous but has no maximum value.
- Any continuous function from $[0,1]$ to $\mathbb{R}$ has a maximum and a minimum value.
- $\int_{[a,b]} f(x)\,dx$ always exists because f is continuous and $[a,b]$ is compact.
- $\int_{[a,b)} f(x)\,dx$ might not exist

Connectedness

<u>def</u>: A space X is <u>connected</u> if the only sets which are both open and closed are X and $\emptyset$.

<u>Theorem</u>: The image of a connected set is connected.

<u>example</u>: Suppose X is compact and connected and $\varphi: X \to \mathbb{R}$ is continuous. Then $\varphi[X] = [\min, \max]$ for some $\min, \max \in \mathbb{R}$.

<u>example</u>: Are $\mathbb{R}$ and $\mathbb{R} \times \mathbb{R}$ isomorphic as topological spaces?