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Group Theory

<u>def</u>: A <u>group</u> is a set G with a function $G \times G \to G$ (called "group multiplication") which is

a) Associative: $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

b) Has identity: $e \cdot g = g \cdot e = g$

c) Has inverses: $g g^{-1} = g^{-1} g = e$

<u>Examples</u>: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^*$, $\mathbb{C}$, $\mathbb{H}$, $\mathrm{Perm}(X)$, $GL_n(\mathbb{R})$, $SU(2)$.

For any object in any category:

$$\mathrm{Aut}(A) \equiv \{\text{isomorphisms from } A \text{ to } A\}$$

is a group with composition as the group multiplication.
hw: Prove it

<u>def</u>: A group is said to be <u>Abelian</u> if it's group multiplication is commutative.

Conventionally, a "+" is often used for group multiplication in an Abelian group.

What can we say about the special group elements?

- There can only be one identity
- Inverses are unique ($\Leftrightarrow$ "-1" is a function)
- $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$ hw: prove these.

From category theory, we know that the morphism in the category of groups is essential.

$$G \xrightarrow{\varphi} H$$

... is a <u>group homomorphism</u> if

$$\varphi(g_1 g_2) = \varphi(g_1)\cdot\varphi(g_2),$$

<u>Does this make a category</u> ?

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\varphi"'] & H \arrow[r, "\psi"] & K \end{tikzcd}$$

$\psi\circ\varphi(g_1 g_2) = \psi(\varphi(g_1 g_2)) = \psi(\varphi(g_1)\,\varphi(g_2)) = \psi(\varphi(g_1))\cdot\psi(\varphi(g_2))$

$\Rightarrow \psi\circ\varphi(g_1 g_2) = \psi\circ\varphi(g_1)\cdot\psi\circ\varphi(g_2) \Rightarrow \psi\circ\varphi$ is also a group homomorphism. Since identity maps are also morphisms, we have a category.

How do morphisms interact with the special group elements?

$$G \xrightarrow{\varphi} H$$

Nicely:

- $\varphi(e_G) = e_H$
- $\varphi(g^{-1}) = \varphi(g)^{-1}$

Proof: hw

Naturally occurring morphisms:

$$\left.\begin{array}{l} L_g : x \longmapsto gx \\ R_g : x \longmapsto xg \end{array}\right\}$$ Invertible functions $G \to G$ but not generally morphisms

hw: prove that L_g, R_g are permutations

$C_g : x \longmapsto g^{-1} x g$ ⟵ A real morphism also, since $C_g(x \cdot y) = g^{-1} x y g = g^{-1} x g g^{-1} y g = C_g(x) \cdot C_g(y)$.

$$\{ C_g : g \in G \} \hookrightarrow \operatorname{Aut} G$$

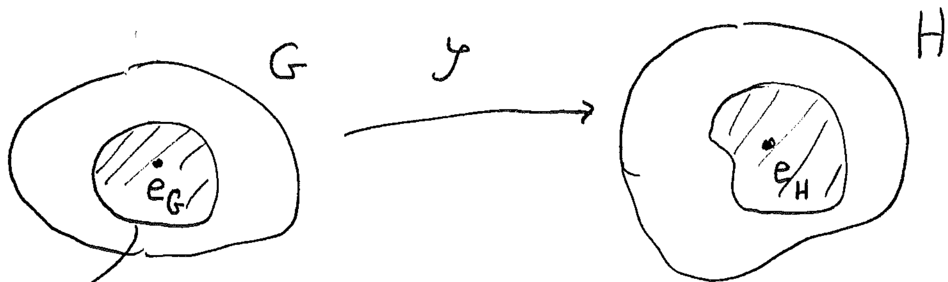
is called the group of inner automorphisms.

Subgroups

def: A subset H of a group G is called a *subgroup of G* if it is a group with the group multiplication from G.

How do we get some?

- Each group homomorphism gives ~~E~~ you two:



$\mathrm{Ker}\,\varphi \equiv \{ g \in G : \varphi(g) = e_H \}$ $\quad$ $\mathrm{Im}\,\varphi \equiv \{ \varphi(g) : g \in G \}$

Both are subgroups.

Proof: hw

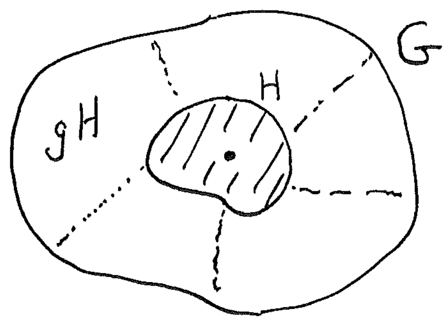
How can a subgroup interact with individual group elements?

If H is a subgroup of G,

$$g, H \mapsto ?$$

Can try

$$gH \equiv \{gh : h \in H\}$$ ...called a "left coset".



Thm: Left cosets cover G without overlapping.

Proof: Suppose $x \in g_1 H$, $x \in g_2 H$.

$\Rightarrow x = g_1 h_1$ for some $h_1 \in H$,

$\Rightarrow x = g_2 h_2$ for some $h_2 \in H$

$\Rightarrow g_1 H = g_2 h_2 h_1^{-1} H = g_2 H$.

Here's a really good question:

Can the left cosets $g_1 H, g_2 H, \ldots$ be made into a group themselves?

Answer: Sometimes.

def: The set of left cosets is denoted "G/H".

There is an easy way to define coset multiplication

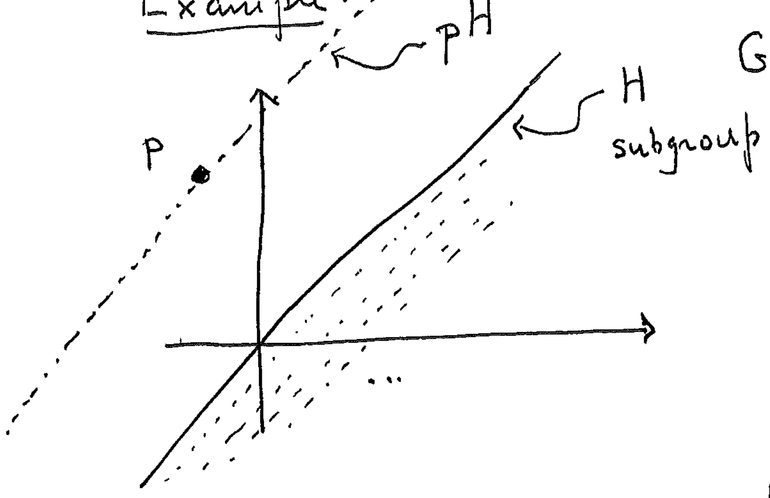
$$(g_1 H) \times (g_2 H) \equiv g_1 H g_2 H$$

The problem is that $g_1 H g_2 H$ might not be a left coset. If, however, we knew that $H g_2 = g_2 H$, then

$$g_1 H g_2 H = g_1 g_2 H H = g_1 g_2 H$$

is another coset. You can easily check that if H has the property ($Hg = gH$ for all $g \in G$), then G/H with multiplication as defined is another group, called the <u>quotient group</u> G/H.

<u>Example</u>:



$G = \mathbb{R} \times \mathbb{R}$

def: A subgroup H with $gH = Hg$ for all $g \in G$ is called a <u>normal</u> subgroup.

As promised, G/H covers the plane without overlapping.

<u>Why didn't we just define $(g_1 H) \times (g_2 H) \equiv g_1 g_2 H$?</u>

Answer: An important homework problem.

Note that the condition that H is a normal subgroup of G ($gH = Hg$ for all $g \in G$) is equivalent to $gHg^{-1} = H$ for all $g \in G$.

<u>Nice fact:</u> For any $G \xrightarrow{\varphi} H$, $\mathrm{Ker}\,\varphi$ is a normal subgroup of G.

<u>Proof:</u> Let $x \in g(\mathrm{Ker}\,\varphi)g^{-1}$. Then $x = gkg^{-1}$ for some $k \in \mathrm{Ker}\,\varphi$. $\Rightarrow \varphi(x) = \varphi(g)\varphi(k)\varphi(g^{-1}) = e_H$
$\Rightarrow x \in \mathrm{Ker}\,\varphi \Rightarrow g(\mathrm{Ker}\,\varphi)g = \mathrm{Ker}\,\varphi \Rightarrow \mathrm{Ker}\,\varphi$ is normal.

<u>Good news:</u> $G \xrightarrow{\varphi} H$

monomorphism $\Leftrightarrow$ 1-1 morphism $\Leftrightarrow$ $\mathrm{Ker}\,\varphi = \{e\}$
epimorphism $\Leftrightarrow$ onto morphism $\Leftrightarrow$ $\mathrm{Im}\,\varphi = H$
isomorphism $\Leftrightarrow$ 1-1 onto morphism $\Leftrightarrow$ $\mathrm{Ker}\,\varphi = \{e\}$ and $\mathrm{Im}\,\varphi = H$

<u>Bad news:</u> Proofs are harder than you might guess. epi is done in Geroch. Do mono for a homework problem.

Also nice: All left cosets are isomorphic as sets.

The product in group.

$$\begin{tikzcd} G & G \times H \arrow[l, "\alpha"'] \arrow[r, "\beta"] & H \\ & X \arrow[ul, "\varphi"] \arrow[u, "\gamma"'] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Guess: $G \times H$ with $(g,h)\cdot(g',h') \equiv (gg', hh')$.
This is a group.

From the category of sets, we already know that the diagram commutes for a unique <u>function</u>

$$\gamma : x \mapsto (\varphi(x), \psi(x))$$

We just have to check that γ is a group homomorphism.

$$\gamma(x \cdot y) = (\varphi(x\cdot y), \psi(x \cdot y)) = (\varphi(x), \psi(x)) \cdot (\varphi(y), \psi(y)) \checkmark.$$

By abstract nonsense, the product is unique!

As in the category of sets, every group homomorphism has an isomorphism of groups hidden inside:

$$\begin{tikzcd} G \arrow[r, "epi"] \arrow[rrr, bend left, "\varphi"] & G/\mathrm{Ker}\,\varphi \arrow[r, "iso"] & \mathrm{Im}\,\varphi \arrow[r, "mono"] & H \\ g \arrow[r, mapsto] & g\,\mathrm{Ker}\,\varphi \arrow[r, mapsto] & \varphi(g) \arrow[r, mapsto] & \varphi(g) \end{tikzcd}$$

This is usually the best way to show that two groups are isomorphic.

Quiz: What do these have in common?

$\exp(x)$

$\lambda(f) \equiv \int_{[a,b]} f(x)\,dx$

$\det(A)$

$\mathrm{Re}(z)$

$\mathrm{abs}(x)$

$\log(x)$

sign of a permutation

$\mathrm{fraction}(x)$

$\mathrm{sign}(k)$

Just having a single morphism can mean a lot:

For example, just the fact that

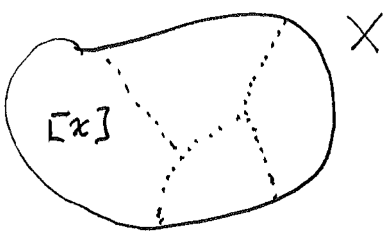
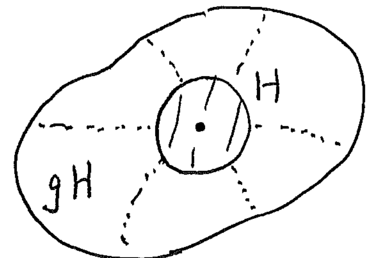
$$\begin{tikzcd} \mathcal{O}(3) \arrow[r, "\det"] \arrow[rr, bend right, "\mathrm{sign} \circ \det"'] & \mathbb{R} \arrow[r, "\mathrm{sign}"] & \{+1, -1\} \end{tikzcd}$$

is an epimorphism let's us conclude the following:

- $SO(3)$ is a subgroup of $\mathcal{O}(3)$
- $SO(3)$ is normal
- $SO(3)$ has exactly one coset "$SO(3)^-$"
- Any element of $\mathcal{O}(3)$ can be written as $\sigma \cdot z$ where σ is a fixed element of $SO(3)^-$ and $z \in SO(3)$.
- $\mathcal{O}(3) = \sigma \cdot SO(3)$
- $\mathcal{O}(3) / SO(3) \cong \{+1, -1\}$
- $SO(3)^- \cdot SO(3)^- = SO(3)$

Homework (Extra): Prove that the sign of a permutation is either even or odd but not both.

Comparison: Sets / Groups

The category of Sets	The category of Groups
Set	Group
function	group homomorphism
	
Equivalence classes partition X without overlapping	Left cosets partition G without overlapping.
X/E is a set	G/H is a group if H is a normal subgroup.
$X \xleftarrow{\alpha} X \times Y \xrightarrow{\beta} Y$	$G \xleftarrow{\alpha} G \times H \xrightarrow{\beta} H$
Products exist, are cartesian.	Products exist with component-wise group multiplication
$X \xrightarrow{\text{epi}} X/E \xrightarrow{\text{iso}} \operatorname{Im}\varphi \xrightarrow{\text{mono}} Y$, $\quad X \xrightarrow{\varphi} Y$	$G \xrightarrow{\text{epi}} G/\operatorname{Ker}\varphi \xrightarrow{\text{iso}} \operatorname{Im}\varphi \xrightarrow{\text{mono}} H$, $\quad G \xrightarrow{\varphi} H$
Every morphism has an isomorphism inside.	Every morphism has an isomorphism inside.