The Quaternions, SU(2), SO(3)

From the ring notes, we defined the quaternions ($\mathbb{H}$) as an algebra on $\mathbb{C}\times\mathbb{C}$ where

$$(a_1, b_1) + (a_2, b_2) \equiv (a_1 + a_2,\ b_1 + b_2)$$

$$(a_1, b_1)\cdot(a_2, b_2) \equiv (a_1 a_2 - b_2 b_1^*,\ a_1^* b_2 + a_2 b_1)$$

$$(a, b)^* \equiv (a^*, -b)$$

This is a non-commutative ring with $(1,0)\cdot(a,b) = (a,b) = (a,b)\cdot(0,1)$. Crucially, $\mathbb{H}$ has a <u>square norm</u>

$$\| a, b \| \equiv a a^* + b b^* \geq 0$$

Since $q q^* = (\|q\|, 0)$, every non zero $q \in \mathbb{H}$ has an inverse $q^{-1} = q^*/\|q\|$.

Note that $\mathbb{H}$ has a representation within complex 2×2 matrices as

$$q = (a,b) \overset{\Psi}{\longleftrightarrow} \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \times \Big(SU(2) \Big)$$

with $\|q\| = \det\begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix} = \det(\Psi(q))$. This is obviously a set isomorphism. The main point is that $\Psi(qq') = \Psi(q)\cdot\Psi(q')$ so matrix multiplication can do the work for us.

From the matrix representation, we have $\|q \cdot r\| = \|q\| \, \|r\|$ (using determinants) and the fact that quaternion multiplication is associative.

- These are very special properties shared by only a few rings ($\mathbb{R}, \mathbb{C}, \mathbb{H}$).

Besides the usual presentation of quaternions on $\mathbb{R}^4$ and on $\mathbb{C} \times \mathbb{C}$ as we have just done, we can also use $\mathbb{R} \times \mathbb{R}^3$:

$$q = (a, A) \in \mathbb{R} \times \mathbb{R}^3$$

vector cross product

$$(a, A) \cdot (b, B) = (ab - A \cdot B, \; aB + bA + A \times B)$$

vector dot product

$$(a, A)^* = (a, -A).$$

The quaternions $(0, A)$ [a.k.a. $q^* = -q$] are called <u>pure</u> quaternions.

Let $C_q(x) \equiv q^{-1} x q$ be conjugation by a nonzero quaternion q. Since

$C_q(x)$ is pure iff x is pure,

$C_q : \text{Pure} \to \text{Pure}$ is onto, linear and norm-preserving and is thus an element of $O(3)$. Since $\mathbb{H} - \{0\}$ is path connected to the identity, C_q must actually be in $SO(3)$. We thus have a map

$$\Psi : q \mapsto C_q \in SO(3)$$

from the group of nonzero quaternions onto $SO(3)$. $q = (a, A)$ is in the kernel of Ψ iff $(a, A)(0, P) = (0, P)(a, A)$ for all $P \in \mathbb{R}^3$. $\Leftrightarrow (-A \cdot P, aP + A \times P) = (-A \cdot P, aP + P \times A)$ for all $P \in \mathbb{R}^3 \Leftrightarrow P = 0 \Leftrightarrow q = q^*$. Since $C_{\lambda q} = C_q$ for $\lambda \neq 0$, we might as well restrict Ψ to the unit quaternions (S^3). Thus we have

$$\Psi : S^3 \twoheadrightarrow SO(3)$$

$\operatorname{Ker} \Psi = \{+1, -1\}$. Therefore

$$S^3 / \{1, -1\} \cong SO(3) \cong SU(2) / \{1, -1\}.$$

More explicitly, a unit quaternion

$$q = (\cos\theta,\ u \cdot \sin\theta) \qquad u \in \mathbb{R}^3 \quad \|u\| = 1$$

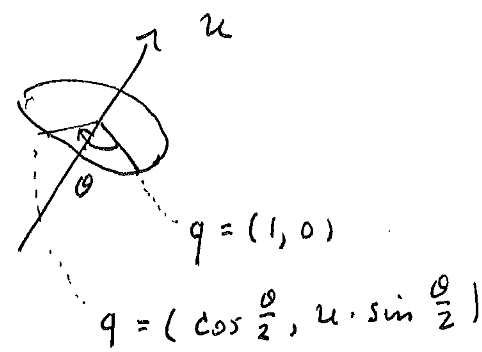
rotates a vector $P \in \mathbb{R}^3$ to

$$q^{-1}(0, P)\, q$$

which is a rotation by 2θ about an axis u.

Obviously, q and $-q$ give the same rotation.

Why not just chose a sign then?



- Because you can't do that continuously.. If you choose $q = (1,0)$ to represent the identity, if you rotate $\theta \to 2\pi$, you end up at $(-1, 0)$ going around once.

LIE GROUPS

... we don't have time for these,
but I can't resist saying something.

This is

- One of the most important areas for applications, especially in physics,
- Everything works spectacularly well mathematically

<u>def</u>: A <u>Lie group</u> is both a group and a smooth manifold where group multiplication is smooth.

<u>Examples</u>: $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $Sp(n,\mathbb{R})$, $S^1 \times S^1 \times \cdots \times S^1 = \mathbb{T}^k$

$\left.\begin{array}{l} O(n) \\ U(n) \end{array}\right\}$ compact

$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ "nilpotent"

$\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ "solvable"

$SU(2) \cong$ unit quaternions

$SO(n)$

Rigid motions in $\mathbb{R}^n$

G_2, F_4, E_6, E_7, E_8 "exceptional"

The Lorentz, Poincaré groups in Physics

In physics, a "particle" is an irreducible representation of the Poincaré group.

A Lie group is a smooth manifold and must therefore have smoothly related charts as we defined last time.

However:

- Any mere topological group which is locally isomorphic to $\mathbb{R}^n$ is automatically a full Lie group with smooth charts and smooth group multiplication.

  (Montgomery and Zippin, 1952).

- Suppose H is a subgroup of Lie group G just in the group theory sense. There is no obvious reason for H to have nice extra properties. However, if H is also closed in the topology of G, then H is automatically a full Lie group.

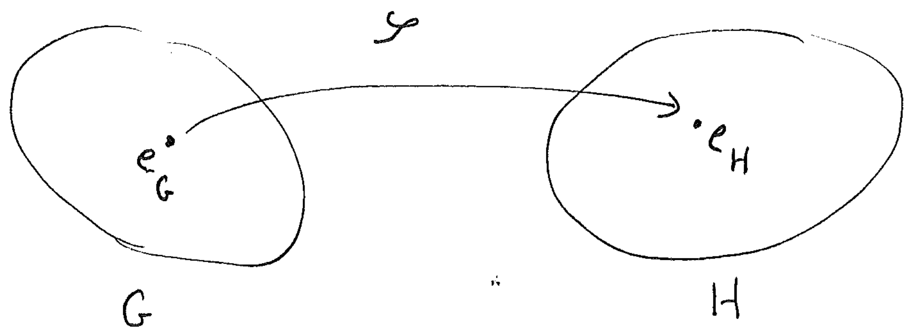
A morphism $G \xrightarrow{\varphi} H$ in the <u>category of Lie groups</u> is a smooth group homomorphism.

- If $\varphi : G \to H$ is merely a continuous group homomorphism, then φ <u>is automatically C^∞</u>.
- Unlike in the case of manifolds,

$$G \to G/\operatorname{Ker}\varphi \longrightarrow \operatorname{Im}\varphi \to H$$

works perfectly. $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, $G/\operatorname{Ker}\varphi$ are all Lie groups.

<u>example</u>: $GL_n(\mathbb{R})$ is a Lie group. $\det : GL_n(\mathbb{R}) \to \mathbb{R}$ is a continuous group homomorphism. Therefore, $\det$ is smooth and $\det^{-1}[1] = SL_n(\mathbb{R})$ is also a Lie group.



<u>Fact</u>: If G is connected, then any open neighborhood of the identity generates the whole group.

<u>Proof</u>: Let $\mathcal{O} \ni e$. Let H be the subgroup generated by $\mathcal{O}$. $H = \bigcup_{h \in H} h \cdot \mathcal{O}$, so H is open. Thus, every coset gH of H is also open, so H is closed as well. Since G is connected, $H = G$.

- φ is thus determined entirely by what it does on any open neighborhood of e, for

$$\varphi(g) = \varphi(\mathcal{O}_1^{\pm} \cdot \mathcal{O}_2^{\pm} \cdot \ldots \cdot \mathcal{O}_k^{\pm}) \qquad \mathcal{O}_i \in \mathcal{O}.$$

φ is, in fact, determined by its differential $d\varphi_e$ at the identity.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, dashed] & H \arrow[d, dashed] \\ TG_e \arrow[r, "d\varphi_e"] & TH_e \end{tikzcd}$$

Connected, simply connected Lie groups.

equivalence of categories

Lie Algebras

"One parameter subgroups"

$$T_e G \cong \{\text{group homomorphisms } \mathbb{R} \to G\}$$

$$\exp : T_e G \to G$$

exp is onto if G is compact. In $SU(2)$, for example, every group element is $\exp(v)$ for some $v \in T_e G$.

<u>ex:</u> $G = GL_n(\mathbb{R})$

$T_e G = M_n(\mathbb{R})$

$$\exp : M_n(\mathbb{R}) \to GL_n(\mathbb{R})$$

$n=1$: $\exp : \mathbb{R} \to \mathbb{R}^{\neq 0}$

- Integration on Lie groups

Not only are all Lie groups orientable, there is a unique left-invariant group integral

$$\int_G f(g) = \int_G f(hg) \quad \text{for all } h,\ f : G \to \mathbb{R}.$$