Vector Space/Topology Exercises

July 19, 2013

1. Show that if area(v, w) on page 4 of the second set of vector space notes is bilinear and if area$(v, v) = 0$, then area is anti-symmetric.

2. Prove that $\det(\psi \circ \varphi) = \det(\psi) \cdot \det(\varphi)$ as claimed on page 5 of the notes.

3. Prove all of the claims on page 7 of the notes.

4. Prove that eigenvectors with different eigenvalues are independent.

5. In a finite dimensional vector space over a field F, prove that $\lambda \in F$ is an eigenvector of operator L if and only if $\det(L - \lambda) = 0$.

6. Read mini-chapters 25, 26, 27, 28, 29 and 30 in Geroch.

7. Consider $A \xrightarrow{\varphi} B$. Does $A \subset A'$ imply $\varphi[A] \subset \varphi[A']$? Does $B \subset B'$ imply $\varphi^{-1}[B] \subset \varphi^{-1}[B']$?

8. Prove that if X is a topological space, X and the empty set are closed, that arbitrary intersections of closed sets are closed and that the union of two closed sets is closed.

9. Prove that if objects are topological spaces and morphisms are continous mappings, that this is a category.

10. Prove that the open sets of a metric space is a topology on the metric space.

11. Prove that every open subset of the real line is a union of open intervals.

12. Prove that every subset of the real line is an intersection of open sets (it's easier than it sounds).

13. Let $\mathbf{R} \overset{\varphi}{\to} \mathbf{R}^2$ be given by $\varphi(r) = (\cos r, \sin r)$. Show that φ is continuous.

14. Find an isomorphism from the subspace $(0, 1)$ of the real line to $\mathbf{R}$.

15. If A is a subset of topological space X, show that in the standard inherited topology, closed sets in A are sets of the form $A \cap C$ with C closed in X.

16. Suppose that A is a collection of subsets of a set X which is closed under pairwise intersection and which includes both X and the empty set. Prove that arbitrary unions of sets in A is the topology generated by A.

17. The direct product of topological spaces X and Y is a certain topology on the cartesian product $X \times Y$. This space is defined as the topology generated by sets $\alpha^{-1}[O_X]$ and $\beta^{-1}[O_Y]$ where α and β are the standard projections and O_X and O_Y are open sets in X and Y respectively. Geroch then notes that this topology is arbitrary unions of sets $\alpha^{-1}[O_X] \cap \beta^{-1}[O_Y]$. Why is this true?

18. For navigation purposes, it would be convenient to have a continuous "smooth" invertible mapping from the surface of the earth to the plane. Is this possible? If not, why not?

19. Let X be a set with the discrete topology. When is X compact?

20. Prove that $\mathbf{R}$ does not have the same topology as $\mathbf{R}^2$ (hint: consider "removing a point from $\mathbf{R}$").

21. Prove that the real line is connected (see Geroch if you get stuck).