Vector Space Exercises I

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1. Prove that $0 \cdot v = 0$ in any vector space.
2. Prove that vector spaces over a fixed field F with linear maps is a category.
3. What is the free vector space on the empty set?
4. Use general abstract nonsense to prove that free constructions are unique in any category.
5. Prove that both the kernel and image of any vector space morphism is a sub-vector space.
6. Find a vector space which is isomorphic to one of it's proper subspaces.
7. Fix a natural number n. Prove that the set of polynomials with degree less than or equal to n with real coefficients is a vector space over the reals.
8. Prove the statement on page 5 of the notes.
9. Prove that every linear map factorizes "with an isomorphism in the middle" as claimed on page 6 of the notes.
10. Suppose $V \overset{\varphi}{\to} W$ is a linear map. Use the fact that complementary subspaces exist to prove that V is isomorphic to $\mathrm{Ker}(\varphi) \oplus \mathrm{Im}(\varphi)$. Note that this implies the "rank-nullity" theorem in the finite dimensional case. Compare how easily you get this result with wikipedia or a typical linear algebra book.

11. Prove that in the category of vector spaces over a fixed field F, monomorphisms preserve independence, epimorphisms preserve span and isomorphisms preserve bases as claimed on page 9 of the notes.

12. Fill out the details of the proof that every vector space has a basis on page 10. Why are the three bulleted items true?