Homework 2

June 27, 2013

1. Prove that a group has only one identity, that inverses are unique, that if $G \xrightarrow{\varphi} H$ is a morphism that $\varphi(e_G) = e_H$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$. Prove that the composition of two group homomorphisms is still a group homomorphism.

2. Are the reals with real multiplication a group? Show that the group of positive real numbers with multiplication as the group multiplication is isomorphic to the group of all real numbers with addition as the group multiplication.

3. The integers ($\mathbf{Z}$) are a group with addition as the group operation. Show that all multiples of 6 are a normal subgroup. What is $\mathbf{Z}/6\mathbf{Z}$?

4. Prove that every group is the subgroup of the permutation group on a set.

5. If H is a normal subgroup of G, why is G/H called a "quotient"?

6. Is the set intersection of subgroups a subgroup? Is the union of subgroups a subgroup?

7. Let $A, B, C, \ldots$ be subsets of a group G with $A \cdot B \equiv \{a \cdot b | a \in A \text{ and } b \in B\}$. Is this a group?

8. Prove that if a subgroup of a group has exactly two left cosets, the subgroup is normal.

9. Prove that the kernel of any group homomorphism is normal.

10. The set of elements in a group which commute with every group element is called the *center* of the group. Prove that the center is always a normal, Abelian subgroup.

11. Prove that $G \stackrel{\varphi}{\to} H$ is one-to-one if and only if $\mathrm{Ker}(\varphi)$ contains only e, and onto if and only if $\mathrm{Im}(\varphi) = H$.

12. Prove that a group morphism is a monomorphism if and only if it is a 1-1 group homomorphism.

13. In the group of permutations of a finite set, group elements can be either even permutations or odd permutations, but not both. Prove this.

14. Is there a direct sum in the category of groups? If so, what is it?.

15. Find the hidden isomorphism inside $abs : \mathbf{R}^{nonzero} \to \mathbf{R}^{positive}$ to show that $\mathbf{R^{nonzero}/\{1, -1\} \cong R^{positive}}$. If you know what a quaternion is, show that conjugation of pure quaternions by unit quaternions gives a group homomorphism from the three sphere to $SO(3)$. Show that the hidden isomorphism in this case gives the famous isomorphism $SU(2)/\{1, -1\} \cong SO(3)$.