Group Theory Exercises

3. $\mathbb{Z}/6\mathbb{Z} \equiv$ "$\mathbb{Z}_6$" are the integers mod 6 :

$\{0+6\mathbb{Z}, 1+6\mathbb{Z}, 2+6\mathbb{Z}, 3+6\mathbb{Z}, 4+6\mathbb{Z}, 5+6\mathbb{Z}\}$

with, e.g. $(5+6\mathbb{Z})+(1+6\mathbb{Z}) = (0+6\mathbb{Z})$.

4. For any group G, $g \mapsto L_g$ is a group monomorphism from G to $\mathrm{Perm}(G)$.

5. I guess because when G is a finite group $|G/H| = |G|/|H|$.

6. Yes. No.

7. Inverses are lacking, so no.

8. Since $gH \longleftrightarrow Hg$ is a bijection between left and right cosets, there are also two right cosets. Since both left and right cosets cover the group $gH = Hg$ for all g and H is normal.

9. Let $G \xrightarrow{\varphi} H$. $x \in g^{-1}\,\mathrm{Ker}\,\varphi\, g \Rightarrow x = g^{-1}kg$ for some $k \in \mathrm{Ker}\,\varphi$ $\Rightarrow \varphi(x) = e_H \Rightarrow x \in \mathrm{Ker}\,\varphi$. Similarly, if $k \in \mathrm{Ker}\,\varphi$, $gkg^{-1} \in \mathrm{Ker}\,\varphi \Rightarrow k \in g^{-1}\,\mathrm{Ker}\,\varphi\, g$.

10. If a, b are in the center of a group, then $(a\cdot b)\cdot x = axb = xab$ $\Rightarrow a\cdot b$ is in the center. Likewise for a^{-1}. If C is the center, then $xC = Cx$, so C is an Abelian normal subgroup.

11. Suppose $G \xrightarrow{\varphi} H$ and $\ker \varphi = \{e_G\}$. If $\varphi(g_1) = \varphi(g_2)$, $\varphi(g_1 g_2^{-1}) = e_H \Rightarrow g_1 = g_2 \Rightarrow \varphi$ is a monomorphism.

12. Suppose that $G \xrightarrow{\varphi} H$ is a 1-1 group homomorphism. Then it is a monomorphism in the category of sets and, therefore, also in the category of groups. Conversely, suppose that φ is a group monomorphism and suppose $\varphi(g_1) = \varphi(g_2)$. Note that

$$X \equiv \{ (g_1 g_2^{-1})^K : K \in \mathbb{Z} \}$$

is a subgroup of G and let

$$\begin{tikzcd} X \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & G \arrow[r, "\varphi"] & H \end{tikzcd}$$

$\alpha : g \mapsto g$, $\beta : g \mapsto e_G \Rightarrow \varphi \circ \alpha = \varphi \circ \beta \Rightarrow \alpha = \beta$
$\Rightarrow g_1 = g_2 \Rightarrow \varphi$ is a 1-1 function.

13. See separate note

14. Suppose that G and H are groups. Let $G \cup_d H \xrightarrow{\lambda} F$ be the free group on $G \cup_d H$ as a set. Consider the following:

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d, "i_G"'] & G \cup_d H \arrow[d, "\lambda"] \arrow[dd, bend left, "\gamma"] & H \arrow[l, "\beta"'] \arrow[d, "i_H"] \\
G \arrow[r, "\lambda \circ \alpha"] \arrow[dr, bend right, "\varphi"'] & F \arrow[d, "\gamma'"'] & H \arrow[l, "\lambda \circ \beta"'] \arrow[dl, bend left, "\psi"] \\
 & X &
\end{tikzcd}$$

Given any X, φ, ψ, the set direct sum guaranteer that the outer rim commutes for a unique γ. By the freeness of F, there is a unique γ' such that the triangle $\lambda, \gamma, \gamma''$ commutes.

$$\Rightarrow \gamma' \circ \lambda = \gamma, \quad \gamma \circ \alpha = \varphi, \quad \gamma \circ \beta = \psi$$

$$\Rightarrow \gamma' \circ (\lambda \circ \alpha) = \varphi \qquad \gamma' \circ (\lambda \circ \beta) = \psi$$

$\Rightarrow (F, \lambda \circ \alpha, \lambda \circ \beta)$ is the direct sum in the category of groups.

15. The kernel of $\text{abs}: \mathbb{R}^{\neq 0} \to \mathbb{R}^{>0}$ is $\{1, -1\}$. Since abs is onto, $\mathbb{R}^{\neq 0}/\{1,-1\} \cong \mathbb{R}^{>0}$. For $SU(2)/\{1,-1\} \cong SO(3)$, see the separate note.