

Group Theory Exercises

3. $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$ are the integers mod 6:

$$\{0+6\mathbb{Z}, 1+6\mathbb{Z}, 2+6\mathbb{Z}, 3+6\mathbb{Z}, 4+6\mathbb{Z}, 5+6\mathbb{Z}\}$$

with, e.g. $(5+6\mathbb{Z}) + (1+6\mathbb{Z}) = (0+6\mathbb{Z})$.

4. For any group G , $g \mapsto L_g$ is a group monomorphism from G to $\text{Perm}(G)$.

5. I guess because when G is a finite group $|G/H| = |G|/|H|$.

6. Yes. No.

7. Inverses are lacking, so no.

8. Since $gH \leftrightarrow Hg$ is a bijection between left and right cosets, there are also two right cosets. Since both left and right cosets cover the group $gH = Hg$ for all g and H is normal.

9. Let $G \xrightarrow{f} H$. $x \in g^{-1} \text{Ker } f \Rightarrow x = g^{-1}k$ for some $k \in \text{Ker } f$
 $\Rightarrow f(x) = e_H \Rightarrow x \in \text{Ker } f$. Similarly, if $k \in \text{Ker } f$,
 $gkg^{-1} \in \text{Ker } f \Rightarrow k \in g^{-1} \text{Ker } f$.

10. If a, b are in the center of a group, then $(a \cdot b) \cdot x = a \cdot b \cdot x = a \cdot x \cdot b = x \cdot a \cdot b$
 $\Rightarrow a \cdot b$ is in the center. Likewise for a^{-1} . If C is the center, then $xC = Cx$, so C is an Abelian normal subgroup.

11. Suppose $G \xrightarrow{\varphi} H$ and $\ker \varphi = \{e_G\}$. If $\varphi(g_1) = \varphi(g_2)$,
 $\varphi(g_1 g_2^{-1}) = e_H \Rightarrow g_1 = g_2 \Rightarrow \varphi$ is a monomorphism.

12. Suppose that $G \xrightarrow{\varphi} H$ is a 1-1 group homomorphism.
Then it is a monomorphism in the category of sets
and, therefore, also in the category of groups.

Conversely, suppose that φ is a group
monomorphism and suppose $\varphi(g_1) = \varphi(g_2)$.

Note that

$$X \equiv \{ (g_1 g_2^{-1})^k : k \in \mathbb{Z} \}$$

is a subgroup of G and let

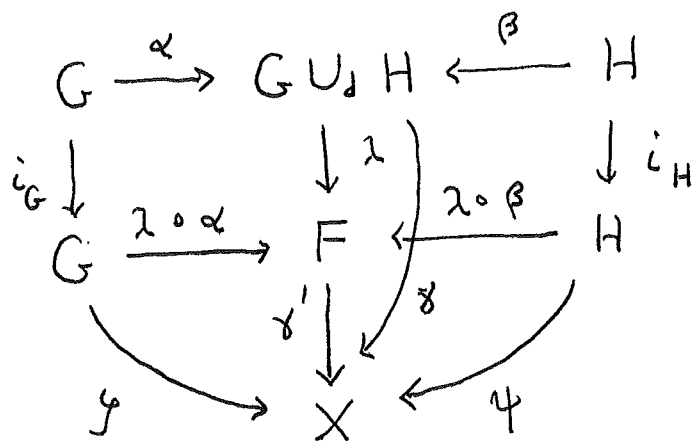
$$X \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} G \xrightarrow{\varphi} H$$

$$\alpha: g \mapsto g, \quad \beta: g \mapsto e_G \Rightarrow \varphi \circ \alpha = \varphi \circ \beta \Rightarrow \alpha = \beta$$

$$\Rightarrow g_1 = g_2 \Rightarrow \varphi \text{ is a 1-1 function.}$$

13. See separate note

14. Suppose that G and H are groups. Let $G \cup H \xrightarrow{\lambda} F$ be the free group on $G \cup H$ as a set. Consider the following:



Given any X, γ, ψ , the set direct sum guarantees that the outer rim commutes for a unique δ .

By the freeness of F , there is a unique δ' such that the triangle λ, δ, δ' commutes.

$$\Rightarrow \delta' \circ \lambda = \delta, \quad \delta \circ \alpha = \gamma, \quad \delta \circ \beta = \psi$$

$$\Rightarrow \delta' \circ (\lambda \circ \alpha) = \gamma \quad \delta' \circ (\lambda \circ \beta) = \psi$$

$\Rightarrow (F, \lambda \circ \alpha, \lambda \circ \beta)$ is the direct sum in

the category of groups.

15. The kernel of $\text{abs}: \mathbb{R}^{\neq 0} \rightarrow \mathbb{R}^{> 0}$ is $\{1, -1\}$.

Since abs is onto, $\mathbb{R}^{\neq 0} / \{1, -1\} \cong \mathbb{R}^{> 0}$. For $\text{SU}(2) / \{1, -1\}$

$\cong \text{SO}(3)$, see the separate note.