Comments mostly about generating sub-objects.

Geroch, page 25:

"As an example of theorem 11, consider the following. Let A be any subset of group G. The intersection of all subgroups of G which contain A itself contains A and is, by theorem 11, also a subgroup of G. This is called the subgroup (of G) <u>generated</u> by the subset A. It is clearly the smallest subgroup of G containing A, in the sense that any other subgroup of G containing A also contains the subgroup generated by A. It is clear that the subgroup of G generated by A consists precisely of the elements of G which can be written as a product of elements of A and their inverses."

$\boxed{\text{STOP}}$ ... Is that really clear?

Let I be the intersection of all subgroups of G containing $A \subset G$.

For $a, b, c, d \in A$, it is indeed clear that any expression such as

$$a\, b^{-1} c^{-1} a\, a\, d$$

must be in I because this element must be in any subgroup containing a, b, c and d. Why, however, is the set of such expressions equal to I? It is because the set of such expressions is itself a group containing a, b, c and d and so

$$\text{Expressions} \subset I$$

and

$$I \subset \text{Expressions}$$

$\Rightarrow I = \text{Expressions}$, so Geroch is correct.

What if we change this as follows:

Let I be the intersection of all <u>normal</u> subgroups containing $A \subseteq G$.

We no longer have $I =$ expressions(A), but this is often done anyway. For example, you may read something like

"Consider the group generated by x, y with $x^{10} = 1$, $y^2 = 1$, $xyxy = 1$."

That actually means the free group on $\{x, y\}$ quotiented by the normal subgroup generated by $\{x^{10}-1,\ y^2-1,\ xyxy-1\}$.

Incidentally, this is the dihedral group D_{10} which we will see later.

... Also, did you notice that I implicitly assumed that the intersection of normal subgroups is normal?

We also defined the equivalence relation generated by a relation $R \subset X \times X$ by

$$Eq(R) \equiv \text{the intersection of all equivalence relations containing } R.$$

To make this less abstract, let $X = \{1, 2, 3, \ldots, n\}$ and let's compute $Eq(R)$ on a computer.

- Algorithm will establish
  $$i \; Eq(R) \; j \iff eq(i) = eq(j)$$
- "Loop invariant"
  $$i \; Eq(R) \; j \Leftarrow eq(i) = eq(j)$$

is easily established by for i in $(1, 2 \ldots, n)$: $eq(i) = i$

A1:

for i in $1, \ldots, n$: $eq(i) = i$

while $eq(i) \neq eq(j)$ for some $(i, j) \in R$:

$eq(i) = eq(j)$

If this terminates, we have $i \; Eq(R) \; j \iff eq(i) = eq(j)$.
It might not terminate as is, but this is easily fixed...

A2:

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for i in 1,...,n: eq(i):= i
while eq(i) ≠ eq(j) for some (i, j) ∈ R:
      if  eq(i) < eq(j): eq(i) := eq(j)
       [] eq(i) > eq(j): eq(j) := eq(i)
      fi
```

This is now guaranteed to terminate because $\sum_{i=1}^{n} eq(i)$ is strictly decreasing and bounded from below.

With a bit more work, this can be made quite efficient [$O(n \log n)$].