Even and Odd Permutations

Let $n = \{1, 2, 3, \ldots, n\}$ and identify permutations with lists in the usual way,

$$3\ 6\ 1\ 5\ 2\ 4 \quad \in \text{Perm}(6)$$

$$1\ 2\ 3\ 4\ 5\ 6 \quad \in \text{Perm}(6) \quad [\text{the identity}]$$

so that the identity perm. is the sorted list.

Let

wrong(i) $\equiv$ the number of list elements to the left of i which "should be" to the right.

e.g.

$$3\ 6\ 1\ 5\ 2\ 4$$

$\uparrow$ wrong$(2) = \#\{3, 6, 5\} = 3$

Let $W \equiv \sum_{i=1}^{n}$ wrong(i).

Note:

- $W = 0$ iff the permutation is the identity
- Every neighbor transposition ("n.t.") changes W by $+1$ or by -1.
- If $W > 0$, W can always be decreased by an n.t.

$\Rightarrow$ Every permutation is a composition of n.t.s.

Suppose that a permutation is both a composition of an even ~~odd~~ number of n.t.s and an odd number of n.t.s. Then

$$1 = \underbrace{t_1 \circ t_2 \circ t_3 \circ \cdots \circ t_k}_{k \text{ odd}}$$

This cannot be, however, because W begins at 0 on the right side and changes by ± 1 an odd number of times.

$\Rightarrow$ For $\pi \in \mathrm{Perm}(n)$,

$$\mathrm{sgn}(\pi) \equiv \begin{cases} +1 & \text{if } \pi \text{ is an even composition of n.t.s} \\ -1 & \text{if } \pi \text{ is an odd composition of n.t.s} \end{cases}$$

is a function.

Note that sgn is a group homomorphism and $\mathrm{sgn}(t) = -1$ for any transposition (not necessarily n.t.).

Even and odd permutations II: The clever Euler-like way:

$$\operatorname{sgn}(\sigma) \equiv \frac{\prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{i<j} (x_i - x_j)} \in \{+1, -1\}$$

That is very clever, because

$$\operatorname{sgn}(\tau \circ \sigma) = \frac{\prod_{i<j} (x_{\tau(\sigma(i))} - x_{\tau(\sigma(j))})}{\prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)})} \, \frac{\prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{i<j} (x_i - x_j)}$$

$$= \operatorname{sgn}(\tau) \cdot \operatorname{sgn}(\sigma).$$

$\Rightarrow \operatorname{sgn} : \operatorname{Perm}(n) \to \{+1, -1\}$

is a group homomorphism.