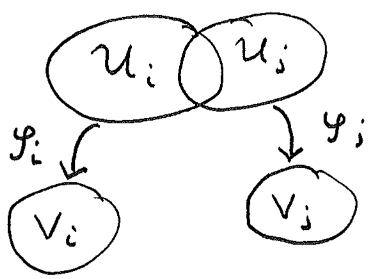
Saul Youssef, 2013

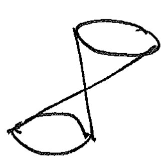
<u>def</u>: An <u>n-dimensional manifold</u> M is a Hausdorff topological space covered by a distinguished set of top-isomorphisms $\varphi_i : \mathcal{U}_i \to V_i$ from open sets $\mathcal{U}_i$ in M to open subsets V_i of $\mathbb{R}^n$.

<u>def</u>: A <u>smooth manifold</u> is a manifold where the induced transitions $\varphi_j \circ \varphi_i^{-1}$ are all smooth.



φ_i are called <u>charts</u>.

<u>Examples</u>: $\mathbb{R}^0$, $\mathbb{R}^n$, $P\mathbb{R}^n$, Torus, Sphere, Space-time, Klein bottle, phase space in classical mechanics, $M_n(\mathbb{R}) \cong \mathbb{R}^n \times \mathbb{R}^n$, $\mathcal{O}(n)$, $S\mathcal{O}(n)$, $SU(n)$, $\mathcal{U}(n)$, other Lie groups, knots, links, algebraic varieties, any open subset of $\mathbb{R}^n$.



← not a manifold (why not?).

Examples:

- $M = \{p\}$ a single point with chart

$$\varphi : p \mapsto 0 \in \mathbb{R}^0$$

- $M = \mathcal{O} \subset \mathbb{R}^n$ any open subset of $\mathbb{R}^n$ with identity map ~~from~~ chart.

- $M \equiv S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$

the unit sphere

$$\mathcal{U}_1 = \{(x,y,z) \in S : x > 0\} \qquad \varphi_1 : (x,y,z) \mapsto (y,z)$$

$$\mathcal{U}_2 = \{(x,y,z) \in S : x < 0\} \qquad \varphi_2 : (x,y,z) \mapsto (y,z)$$

$$\mathcal{U}_3 = \{(x,y,z) \in S : y > 0\} \qquad \varphi_3 : (x,y,z) \mapsto (x,z)$$

$$\mathcal{U}_4 = \{(x,y,z) \in S : y < 0\} \qquad \varphi_4 : (x,y,z) \mapsto (x,z)$$

$$\mathcal{U}_5 = \{(x,y,z) \in S : z > 0\} \qquad \varphi_5 : (x,y,z) \mapsto (x,y)$$

$$\mathcal{U}_6 = \{(x,y,z) \in S : z < 0\} \qquad \varphi_6 : (x,y,z) \mapsto (x,y)$$

$$\varphi_1^{-1}(y,z) = \left((1-y^2-z^2)^{1/2}, y, z\right)$$

$$\varphi_3 \circ \varphi_1^{-1} : (y,z) \mapsto \left((1-y^2-z^2)^{1/2}, z\right) \text{ must be smooth.}$$

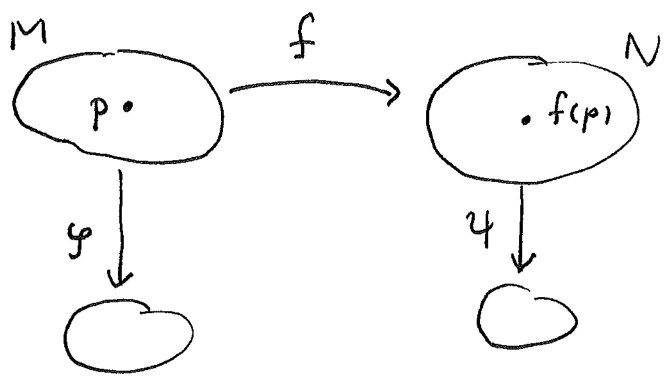
- $M = (\mathbb{R}^n - \{0\})/E \quad vEw$ iff $v = \lambda w$ for some $\lambda \neq 0$ — "Projective space"

$$\mathcal{U}_n = \{[x_1, x_2, \dots, x_n] : x_n \neq 0\} \qquad M \equiv \mathbb{P}\mathbb{R}^n$$

$$\varphi_n : [x_1, x_2, \dots, x_n] \mapsto (x_1/x_n, x_2/x_n, \dots, x_{n-1}/x_n) \in \mathbb{R}^{n-1}$$

Smooth Maps

A continuous map $M \xrightarrow{f} N$ between manifolds is smooth if it is smooth in local charts



i.e. if all such $\psi \circ f \circ \varphi^{-1}$ are smooth in the non-manifold sense.

Given
$$M \xrightarrow{f} N \xrightarrow{g} P$$

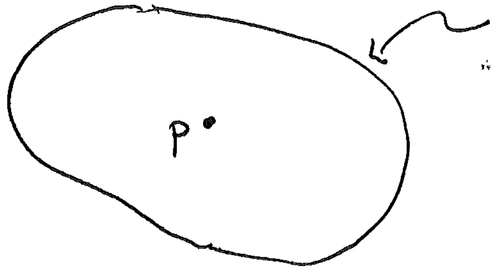
if f and g are smooth, so is $g \circ f$ and so are identities. We thus have the category of smooth manifolds. The category has products

$$\begin{tikzcd} & M \times N \arrow[dl, "\varphi_i"'] \arrow[dr, "\psi_j"] & \\ \bigcirc \subset \mathbb{R}^m & & \bigcirc \subset \mathbb{R}^n \end{tikzcd}$$

$M \times N$ is an $(m+n)$-dimensional manifold with charts $\varphi_i \times \psi_i$.

If M and N have the same dimension, $M \oplus N$ is a manifold with charts $\varphi_i \oplus \psi_i$.

Tangent Space



M, an n-dimensional smooth manifold.

We want to define "the space of possible velocity vectors at p" but we also want to do this "intrinsically" i.e. independent of an equivalent choice of charts.

Consider the set of smooth curves

$$\gamma : I \to M$$

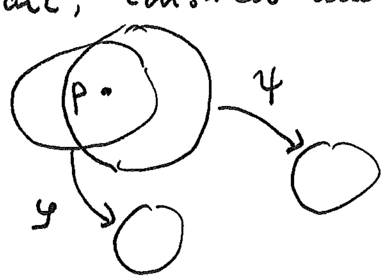
with $0 \in I$ and $\gamma(0) = p$. Let

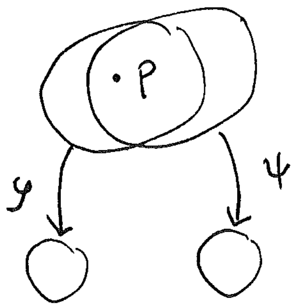
$$\gamma \, E \, \gamma' \text{ iff } d_0(\varphi \circ \gamma) = d_0(\varphi \circ \gamma')$$

for some chart φ.

$$[\gamma] \in TM_p \cong \mathbb{R}^n \text{ the tangent space at } p.$$

To show that $[\gamma]$ is independent of the choice of chart, consider another chart ψ





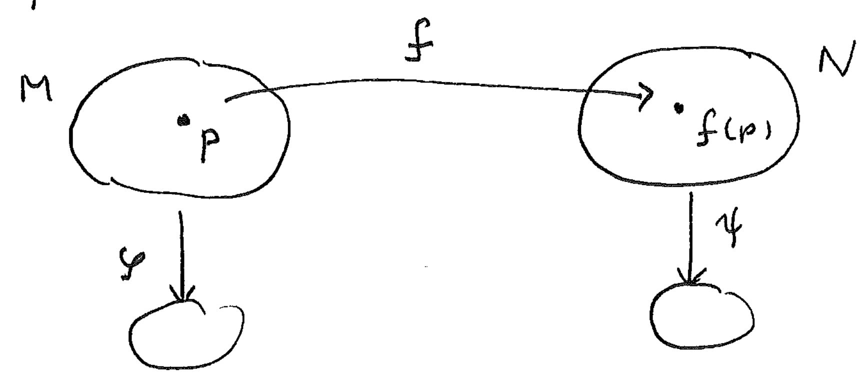
$$d_0(\varphi\circ\gamma) = d_0(\varphi\circ\gamma')$$

$$\Leftrightarrow \quad d_{\varphi(p)}(\psi\circ\varphi^{-1})\circ d_0(\varphi\circ\gamma) = d_{\varphi(p)}(\psi\circ\varphi^{-1})\circ d_0(\varphi\circ\gamma')$$

$$\Leftrightarrow \quad d_0(\psi\circ\varphi^{-1}\circ\varphi\circ\gamma) = d_0(\psi\circ\varphi^{-1}\circ\varphi\circ\gamma')$$ by the chain rule

$$\Leftrightarrow \quad d_0(\psi\circ\gamma) = d_0(\psi\circ\gamma').$$

Using tangent spaces, we can define the differential of a smooth map $M \xrightarrow{f} N$ at $p \in M$.



Let

$$TM_p \ni [\gamma] \xmapsto{\text{"}df_p\text{"}} [f \circ \gamma] \in TN_{f(p)}.$$

To show that this is a function, suppose $\gamma E \gamma'$.

$$\Rightarrow d_0(\varphi \circ \gamma) = d_0(\varphi \circ \gamma')$$

$$\Rightarrow d_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) \circ d_0(\varphi \circ \gamma) = d_{\varphi(p)}(\psi \circ f \circ \varphi^{-1}) \circ d_0(\varphi \circ \gamma')$$

$$\Rightarrow d_0(\psi \circ f \circ \gamma) = d_0(\psi \circ f \circ \gamma')$$

$$\Rightarrow [f \circ \gamma] = [f \circ \gamma']$$

<u>def</u>: If a smooth map between manifolds has a smooth inverse, it is called a <u>diffeomorphism</u>.

Because of the chain rule, the differential is part of the tangent space functor from pointed manifolds to finite dimensional real vector spaces.

$$\begin{array}{ccccc} (M,p) & \xrightarrow{f} & (N,q) & \xrightarrow{g} & (P,r) \\ \downarrow & & \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ TM_p & \xrightarrow{df_p} & TN_q & \xrightarrow{df_q} & TP_r \end{array}$$

By abstract nonsense, $M \cong N \Rightarrow \dim(M) = \dim(N)$.

In other categories, a morphism like

$$M \xrightarrow{f} N$$

gives us new objects "at both ends". For manifolds, the situation is much more delicate and conditional.

<u>Thm</u> (Inverse function): Given a smooth map $f: \mathcal{U} \to \mathbb{R}^n$ from an open subset $\mathcal{U} \subset \mathbb{R}^n$ to $\mathbb{R}^n$, if df_p is iso, then f is a diffeomorphism on some open neighborhood of p.

<u>Proof</u>: Handout

<u>Thm</u>: (Implicit function): Suppose $\mathcal{U} \subset \mathbb{R}^k, V \subset \mathbb{R}^\ell$ are open, $F: \mathcal{U} \times V \to \mathbb{R}^\ell$ is smooth and

$$d_{z_0}(z \mapsto F(x_0, z))$$

is iso at some $z_0 \in \mathcal{U}$ for some $x_0 \in V$. Then there is a

$$(x, z) \longmapsto (x, F(x, z))$$

a smooth diffeomorphism on an open neighborhood of (x_0, z_0).

Example:

$$F(\underbrace{x, y}_{u}, \underbrace{z}_{V}) = x^2 + y^2 + z^2$$

$$d_{z_0}(z \mapsto x_0^2 + y_0^2 + z^2) = 2 z_0 \, dz$$

... which is invertible if we choose $z_0 \neq 0$..

We are then guaranteed some diffeomorphism

$$(x, y, z) \xleftrightarrow{\Psi} (x, y, x^2 + y^2 + z^2)$$

in a neighborhood of $(x_0, y_0, z_0) \ni \mathcal{V}$.

Restrict Ψ to $\mathcal{V} \cap \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$

$$(x, y, z) \xleftrightarrow{\Psi} (x, y, r^2)$$

gives exactly a coordinate chart mapping a piece of the sphere of radius r to $\mathbb{R}^2$.

$$M \xrightarrow{f} N$$

$p \in M$ is a <u>regular point</u> if df_p is epi

$q \in N$ is a <u>regular value</u> if $f^{-1}[p]$ are all regular points

$q \in N$ is a <u>critical value</u> if q is not a regular value.

<u>Claim</u>: If $q \in N$ is a regular value, then $f^{-1}[q]$ is an $(m-n)$-dimensional manifold.

<u>Example</u>:

$$\rho : \mathbb{R}^3 \to \mathbb{R}$$

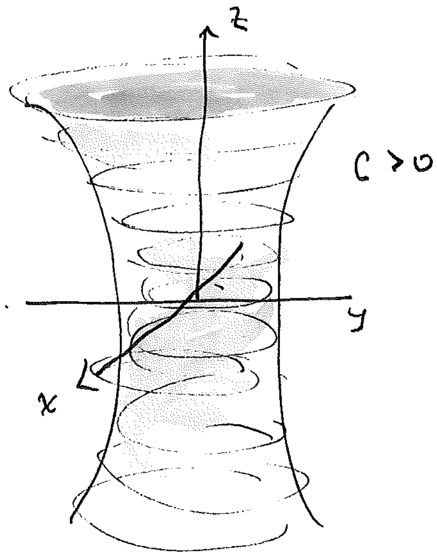
$$\rho(x, y, z) = x^2 + y^2 + z^2$$

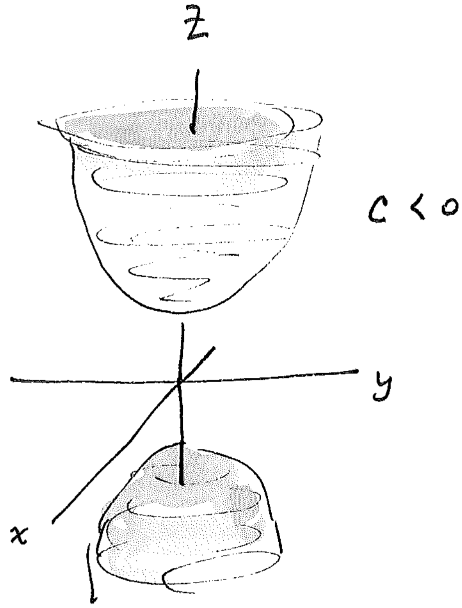
$\rho^{-1}[1] = S^2$ is a submanifold in $\mathbb{R}^3$

Example : $f(x,y,z) = x^2 + y^2 - z^2$

$$df_{(x,y,z)} = 2x\,dx + 2y\,dy - 2z\,dz$$

$(0,0,0)$ is the only critical point, 0 is the only critical value.





$f^{-1}[c]$ is a manifold as long as $c \neq 0$.
As c goes from positive to negative values, $f^{-1}[c]$ changes topology going through the critical point!

On the codomain end, there are also complications.

$$M \xrightarrow{f} N$$

Suppose df_p is mono. (called an <u>immersion</u>).
The inverse function theorem gives us a local diffeomorphism onto the image of f, but this might not be a manifold. e.g.

circle — image is not a manifold.

<u>def</u>: A map is <u>proper</u> if the preimage of a compact set is compact.

<u>def</u>: An injective, proper immersion is called an <u>embedding</u>.

Finally, if f is an embedding, then $f[M]$ is a submanifold of N.

Differential Forms and Vector Fields

Consider $f : \mathcal{U} \to \mathbb{R}$ with $\mathcal{U} \subset \mathbb{R}^n$ a manifold.

$$df_p(h) \equiv \lim_{\lambda \to 0} \frac{f(p+\lambda h) - f(p)}{\lambda} \in T\mathcal{U}_p^*$$

Elements of $T\mathcal{U}_p^*$ are called "one-forms".

For example, with $n = 3$,

$$df_p = \frac{\partial f}{\partial x}(p)\, dx_p + \frac{\partial f}{\partial y}(p)\, dy_p + \frac{\partial f}{\partial z}(p)\, dz_p$$

where $\{dx_p, dy_p, dz_p\}$ is a basis of $T\mathcal{U}_p^*$.

This defines the partial derivatives.

Suppose that $v : \mathcal{U} \to \mathbb{R}^3$ is a smooth vector field.

$$df_p(v_p) = \frac{\partial f}{\partial x} v_x + \frac{\partial f}{\partial y} v_y + \frac{\partial f}{\partial z} v_z$$

$$= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) f \equiv v_p f$$

So v can be thought of as a linear derivation on $C^\infty(\mathcal{U})$. In fact, you can show (handout) that all derivations are vector fields, an intrinsic definition.

The flow of a smooth vector field

Let M be an n-dimensional smooth manifold.

<u>def</u>: A group homomorphism $\mathbb{R} \xrightarrow{\mathcal{F}} \mathrm{Diff}(M)$
is called a <u>flow</u>.

<u>def</u>: A vector field $v: M \to \mathbb{R}^n$ <u>has a flow</u> $\mathcal{F}$ if

$$\frac{d}{dt} \mathcal{F}^t(p) = v\left(\mathcal{F}^t(p)\right)$$

for all $t \in \mathbb{R}$, $p \in M$.

<u>Thm</u>: A smooth vector field on M which vanishes outside of a compact subset of M has a unique flow.

There is a beautiful short proof of this in Milnor's Morse theory book.
I think I will repeat it in a handout.

Example: Hamiltonian flow.

$x : \mathbb{R} \to \mathbb{R}^3$ trajectory

$\varphi : \mathbb{R}^3 \to \mathbb{R}$ potential. $\quad m \dfrac{d^2 x_i}{dt^2} = -(\nabla\varphi)_i$

$m > 0$ mass

$q(t) \equiv x(t)$

Let $\mathcal{H}(p,q) = \dfrac{\langle p, p\rangle}{2m} + \varphi(q)$ $\quad p(t) \equiv m\dot{x}(t)$

$$\frac{d}{dt}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} p_1/m \\ p_2/m \\ p_3/m \\ -(\nabla\varphi)_1 \\ -(\nabla\varphi)_2 \\ -(\nabla\varphi)_3 \end{pmatrix} = \begin{pmatrix} \partial\mathcal{H}/\partial p_1 \\ \partial\mathcal{H}/\partial p_2 \\ \partial\mathcal{H}/\partial p_3 \\ -\partial\mathcal{H}/\partial q_1 \\ -\partial\mathcal{H}/\partial q_2 \\ -\partial\mathcal{H}/\partial q_3 \end{pmatrix}$$ Hamilton's equations

Trajectories in phase space $\mathbb{R}^3 \times \mathbb{R}^3$ are the integral curves of the vector field

$$H = \sum_{i=1}^{3}\left(\frac{\partial\mathcal{H}}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\right)$$

Example: Riemannian manifolds.

$g : TM_p \times TM_p \to \mathbb{R}$ a.k.a. $g \in (TM_p \otimes TM_p)^{*}$

$$\text{Length} \equiv \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\, dt$$

Differential forms

... are easy and very useful.

$$\Lambda^k_p \equiv \underbrace{T\vartheta_p^* \wedge T\vartheta_p^* \wedge \cdots \wedge T\vartheta_p^*}_{k}$$

A smooth choice of Λ^k_p at each $p \in \vartheta$ is called a <u>k-form</u>. Λ^k is the set of k-forms.
Note that Λ^0 is the set of real valued functions on ϑ.

Notation: The general element of Λ^k is

$$\omega = \sum_I{}' a_I \, dx_I$$

I are k choices from $1, 2, \ldots, n$ with $i_1 < i_2 < i_3 < \cdots < i_k$

If
$$\eta = \sum_J{}' b_J \, dx_J \in \Lambda^\ell$$

$$\omega \wedge \eta \equiv \sum_I{}' \sum_J{}' a_I b_J \, dx_I \wedge dx_J \in \Lambda^{k+\ell}.$$

Facts:
$$(\omega \wedge \eta) \wedge \sigma = \omega \wedge (\eta \wedge \sigma)$$
$$(\omega \wedge \eta) = (-1)^{k \cdot \ell} (\eta \wedge \omega)$$

$\omega \in \Lambda^k$
$\eta \in \Lambda^\ell$
$\sigma \in \Lambda^m$

$$\omega \wedge (\eta + \sigma) = \omega \wedge \eta + \omega \wedge \sigma$$
if $\ell = m$.

Pull backs

Both functions $f: \mathcal{V} \to \mathbb{R}$ and one-forms $\omega_p : T\mathcal{V}_p \to \mathbb{R}$ can be "pulled back" through smooth maps.

$$
\begin{tikzcd}
\mathcal{U} \arrow[rr, "f"] \arrow[dr, "f^*\varphi \equiv \varphi\circ f"'] & & \mathcal{V} \arrow[dl, "\varphi"] \\
& \mathbb{R} &
\end{tikzcd}
\qquad
\begin{tikzcd}
T\mathcal{U}_p \arrow[rr, "df_p"] \arrow[dr, "(f^*\omega)_p"'] & & T\mathcal{V}_{f(p)} \arrow[dl, "\omega_{f(p)}"] \\
& \mathbb{R} &
\end{tikzcd}
$$

$$(\varphi^*\omega)_p\,(v) \equiv \omega_{f(p)}(df_p(v))$$

This definition can be extended to k-forms by

$$f^*\omega \equiv \sum_I{}' (f^* a_I)(f^* dx_{i_1}) \wedge (f^* dx_{i_2}) \wedge \cdots \wedge (f^* dx_{i_k})$$

<u>Facts</u>: $f^*(\omega + \eta) = f^*\omega + f^*\eta$

$$f^*(g\cdot\omega) = (f^*g)\cdot(f^*\omega)$$

$$f^*\,\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k = f^*\omega_1 \wedge f^*\omega_2 \wedge \cdots \wedge f^*\omega_k .$$

Example: Pull back the volume element in $\mathbb{R}^2$ $(dx \wedge dy)$ to polar coordinates.

$$f : \mathbb{R}^{\geq 0} \times [0, 2\pi) \to \mathbb{R}^2$$

$$f(r, \theta) = (r\cos\theta, r\sin\theta)$$

$$f^*(dx \wedge dy) = (f^* dx) \wedge (f^* dy) = d(x \circ f) \wedge d(y \circ f)$$

$$= d(r\cos\theta) \wedge d(r\sin\theta)$$

$$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= r\cos^2\theta\, dr \wedge d\theta - r\sin^2\theta\, d\theta \wedge dr = r\, dr \wedge d\theta$$

using $dr \wedge dr = 0$, $d\theta \wedge d\theta = 0$.

$r\, dr \wedge d\theta$ is the volume element in polar coordinates.

In general, the pull back lets you "change coordinates" through any diffeomorphism!

d

The exterior derivative is the single natural differential operator for a smooth manifold.

<u>def</u>: Given $\omega = \sum'_I a_I \, dx_I \in \Lambda^k M$

$$d\omega \equiv \sum'_I da_I \wedge dx_I$$

<u>Facts</u>:

- Is the differential from calculus on Λ^0
- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \quad \omega \in \Lambda^k$
- $d(f^* \omega) = f^* d\omega$ Commutes with pullbacks!
- $d(d\omega) = 0$

The last point relies on $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ for $f \in C^2$ (see handout).

$$\Lambda^0 M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n M$$

Called <u>the</u> <u>de Rham complex</u>.

Exampler: $M = \mathbb{R}^3$

$\Lambda^0 M \ni f$

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz$$

$\Lambda^1 M \ni \omega = A\, dx + B\, dy + C\, dz$

$$d\omega = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx$$

$$+ \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy$$

$\Lambda^2 M \ni \omega = A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy$

$$d\omega = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx \wedge dy \wedge dz$$

$\Lambda^3 M \ni \omega = a \cdot dx \wedge dy \wedge dz$

$$d\omega = 0$$

The Hodge Star in $\mathbb{R}^n$

$$*(dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}) \equiv$$

$$(-1)^{\sigma}(dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_{n-k}})$$

where $i_1 < i_2 < \dots < i_k$

$j_1 < j_2 < \dots < j_{n-k}$

$(i_1, i_2, \dots, i_k, j_1, j_2, \dots, j_{n-k})$ is a permutation of $(1, 2, \dots, n)$, σ is the sign of the permutation.

This defines a linear map $*$ on Λ^k (why?).

- For a general, oriented, n-dimensional manifold with volume form $\nu \in \Lambda^n M$, the Hodge star is defined by

$$\langle u, v \rangle \nu = u \wedge * v$$

where $\langle , \rangle$ is a non-degenerate bilinear form.

- Hodge star requires a choice of volume form and $\langle , \rangle$ for the manifold.

Examples: You only need d.

- The gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$

$$df_p \equiv \langle \cdot , \operatorname{grad} f \rangle$$

where d is either the exterior or usual derivative.

- Divergence of a vector field $v : \mathbb{R}^n \to \mathbb{R}^n$

Let $\omega_p \equiv \langle \cdot , v_p \rangle \in \Lambda^1 \mathbb{R}^n$

$$d(* \omega) = (\operatorname{div} v) \cdot \nu$$

where $\nu = dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$ is the standard volume form.

- Laplacian of $f : \mathbb{R}^n \to \mathbb{R}$

$$d * df = \Delta f \, \nu$$

- Curl of a vector field $v : \mathbb{R}^n \to \mathbb{R}^n$

Let $\omega_p \equiv \langle \cdot , v_p \rangle$

$$\operatorname{curl}(v) = * d\omega \in \Lambda^{n-2}$$

Exercise: Show that $d^2 = 0$ implies $\operatorname{curl}(\operatorname{grad} f) = 0$.

Example: Maxwell's Equations

$$E \equiv E_x\, dx + E_y\, dy + E_z\, dz \in \Lambda^1 \mathbb{R}^4$$

$$B \equiv B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy \in \Lambda^2 \mathbb{R}^4$$

$$F \equiv E \wedge dt + B \in \Lambda^2 \mathbb{R}^4 \quad \text{field strength}$$

$$j = -\rho\, dt + J_x\, dx + J_y\, dy + J_z\, dz \quad \text{current}$$

$*$ $\equiv$ defined by the standard volume form and the Lorentz "inner product".

Maxwell's Equations then are

$$dF = 0 \qquad d*F = 4\pi j$$

Note that if $F = dA$, $dF = d^2 A = 0$ is automatic.

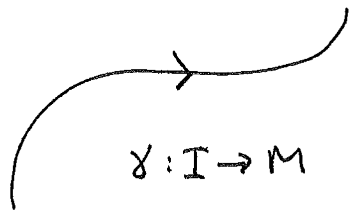
Integration on Manifolds

<u>Example:</u>

M an n-dimensional manifold

$I = [a, b]$

$\omega \in \Lambda^1 M$



$\gamma : I \to M$

Suppose that we want to compute

$$\int_a^b \omega(\gamma(t))\left(\frac{d\gamma}{dt}(t)\right) dt .$$

Note that this is easily done and is equal to

$$\int_I \gamma^* \omega$$

provided that we choose an "orientation" on I to determine the sign.

The idea is to only define integration of n-forms on "oriented" n-dimensional manifolds.

<u>Orientation</u>

Let $\nu, \mu \in \Lambda^n M$ be nowhere vanishing n-forms on n-dimensional manifold M.

$$\nu \, E_\mu \quad \text{iff} \quad \nu = f \cdot \mu$$

for some always positive real function on M.
A choice of equivalence class is an <u>orientation</u> of M. It may be that a manifold has no nowhere vanishing n-forms and thus cannot be oriented.

$$[\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \,]$$

is called the <u>standard orientation</u>.

A smooth map $f : \mathcal{U} \to V$ <u>preserves orientation</u> if $\cancel{\forall p}$ $\det(df_p) > 0$ everywhere.

Under these conditions, the integral of a volume form $\nu = f\, dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$ is defined on n-dimensional manifold M.

$$\int_M \nu = \int_M f\, dx_1 \wedge dx_2 \wedge \dots \wedge dx_n \equiv \int f(x)\, dx_1\, dx_2 \dots dx_n$$

Riemann, any charts

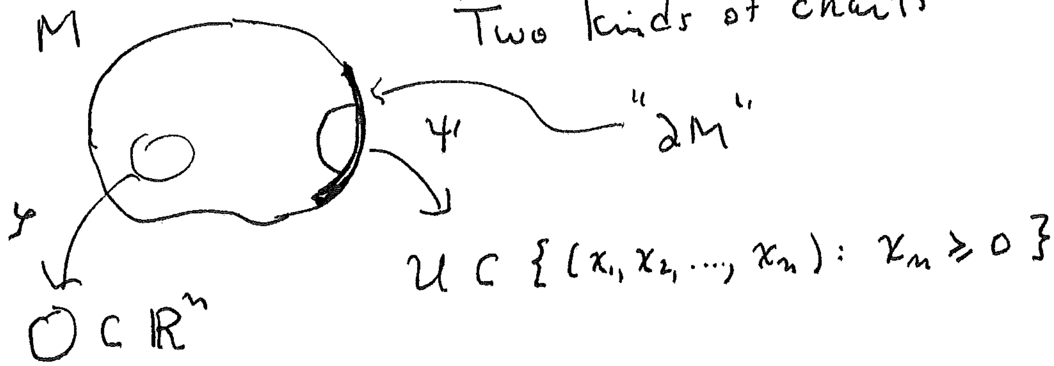
If $M \xrightarrow{f} N$ is an orientation-preserving diffeomorphism,

$$\int_N f^* \nu = \int_M \nu$$

as desired.

To get the general nice theory of integration, we have to introduce "manifolds with boundary".

Two kinds of charts



Tangent spaces remain n-dimensional and defined as before, even on the boundary.

Thm (Stokes): Given an oriented, compact, n-dimensional manifold M with boundary ∂M and inherited orientation. If ω is a smooth $n-1$ form on M, then

$$\int_M d\omega = \int_{\partial M} i^*\omega$$

where $i : \partial M \to M$ is the natural insertion of the boundary ∂M into M.

Example: $M = [a,b]$ is a 1-dimensional manifold with boundary $\partial M = \{a\} \oplus \{b\}$

$$\int_M df = \int_I f = f(b) - f(a)$$

which is "the fundamental theorem of Calculus."