Saul Youssef
CCS/Physics

Summer Math Course

- Start from scratch
- Modernize
- Prove everything
- Blend in applications

Slogan: What I cannot prove, I do not understand.

"Mathematical Physics"
by Robert Geroch
hw 1: Buy this book

Topics:

1. Category Theory
2. Groups and Rings
3. Vector Spaces
4. Topology
5. Analysis
6. Differential Topology
7. Lie Groups
8. More...

Categories ... take a little getting used to.

<u>def</u>: "Objects": $A, B, C, \ldots$

"Morphisms": $\mathrm{mor}(A, B)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Morphisms must compose <u>as if</u> they were functions.

$(h \circ g) \circ f = h \circ (g \circ f)$ Just like functions.

For any object A, there must be a morphism

$$A \xrightarrow{i_A} A$$

such that $f \circ i_A = f$ and $i_A \circ f = f$.

<u>def</u>: <u>Commuting diagrams</u>

ex:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & C \arrow[r, "h"] & D \end{tikzcd}$$

Commutes $\iff$ $\circ$ is associative.

Preview:

Objects	Morphisms
Sets	functions
groups	group homomorphisms
partially ordered sets	order preserving functions
$\mathbb{N}$	$mn(n, m) \equiv \{n \times m \text{ real matrices}\}$
Topolocical Spaces	continuous functions
Manifolds	smooth functions
Lie groups	smooth homomorphisms

$\vdots$

Distinguished morphisms

- If $A \xrightarrow{f} B$ is such that

$$\begin{tikzcd} X \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & A \arrow[r, "f"] & B \end{tikzcd} \Rightarrow \alpha = \beta \quad \text{for all } X, \alpha, \beta$$

then f is a <u>monomorphism</u>.

- If $A \xrightarrow{f} B$ is such that

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & X \end{tikzcd} \Rightarrow \alpha = \beta \quad \text{for all } X, \alpha, \beta$$

then f is an <u>epimorphism</u>.

- Given $A \xrightarrow{f} B$, if a morphism f^{-1} exists such that

$$\begin{tikzcd} A \arrow[loop left, "i_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \arrow[loop right, "i_B"] \end{tikzcd}$$

commutes, then f is an <u>isomorphism</u>.

We then say $A \cong B$.

Example: The category of Sets

objects : sets

morphisms: "functions"

def: A relation f from A to B is a subset of $A \times B$.

Notation: $(a,b) \in f \iff a f b$.

def: A function f from A to B is a relation where $\mathrm{dom}(f) = A$ and afx and $afy \Rightarrow x = y$. see hw problem.

Q: How many functions are there

$$\{\} \xrightarrow{f} \mathbb{R} \;?$$

Ans: Exactly one. hw #2: What is it?

monomorphisms	1-1 functions	"injective"
epimorphisms	onto functions	"surjective"
isomorphisms	both 1-1 and onto	"bijective"

Proof: hw #3

Claim: If $A \xhookrightarrow{f} B$ and $B \xhookrightarrow{g} A$, then $A \cong B$.

Proof: hw #4 [Cantor-Bernstein theorem].

The categorical product

Motivation

$$\begin{tikzcd}
A_1 & A_2 & A_3 & \cdots & A_n \\
 & & X \arrow[ull, "f_1"] \arrow[ul, "f_2"'] \arrow[u, "f_3"'] \arrow[urr, "f_n"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
A_1 & A_2 & A_3 & \cdots & A_n \\
 & & P \arrow[ull, "\alpha_1"] \arrow[ul, "\alpha_2"'] \arrow[u, "\alpha_3"'] \arrow[urr, "\alpha_n"'] & & \\
 & & X \arrow[u, "\gamma"] & &
\end{tikzcd}$$

"Product"
$A_1 \times A_2 \times \cdots \times A_n$

Want $(f_1, f_2, \dots, f_n) \longleftrightarrow \gamma$

<u>def</u>: Given A, B, (P, α, β) is a <u>product</u> of A and B if, for any X, f, g as below,

$$\begin{tikzcd}
A & P \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\
 & X \arrow[u] \arrow[ul, bend left, "f"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

there is a unique γ such that the diagram commutes.

<u>Example</u>: sets $P \equiv A \times B$ is the cartesian product

$\alpha : (a, b) \mapsto a$

$\beta : (a, b) \mapsto b$

$$\begin{tikzcd}
f(x) & (f(x), g(x)) \arrow[l, mapsto, "\alpha"'] \arrow[r, mapsto, "\beta"] & g(x) \\
 & x \arrow[u, mapsto, "\gamma"'] \arrow[ul, mapsto, bend left, "f"] \arrow[ur, mapsto, bend right, "g"'] &
\end{tikzcd}$$

$\gamma(x) \equiv (f(x), g(x))$ ✓

<u>Theorem</u>: Products are unique in any category.

<u>Proof</u>:

$$\begin{tikzcd}
A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\
A \arrow[u, "i_A"] & (A \times B)' \arrow[l, "\alpha'"'] \arrow[r, "\beta'"] \arrow[u, "\gamma"] & B \arrow[u, "i_B"'] \\
A \arrow[u, "i_A"] & A \times B \arrow[l] \arrow[r, "\beta"] \arrow[u, "\gamma'"] & B \arrow[u, "i_B"']
\end{tikzcd}$$

commutes

commutes

$\Rightarrow$ Whole diagram commutes

$\Rightarrow$

$$\begin{tikzcd}
A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\
A \arrow[u, "i_A"] & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] \arrow[u, "\gamma \circ \gamma'"] & B \arrow[u, "i_B"']
\end{tikzcd}$$

commutes

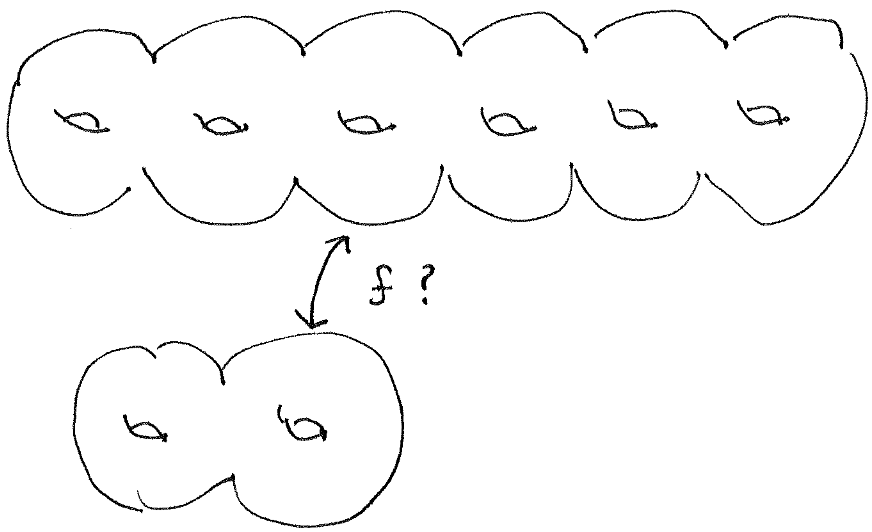
$\Rightarrow \gamma \circ \gamma' = i_{A \times B} \Rightarrow \gamma' \circ \gamma = i_{(A \times B)'} \Rightarrow A \times B \cong (A \times B)'$.

<u>Summary</u>: Products are unique by "abstract nonsense."

hw #5: Re-do these proofs.

hw #6: What is the dual notion categorically [it's called the sum]? What is this in the category of sets?

Intermission



...Back in the category of sets

<u>def</u>: Given a set X, a relation $\leq \subset X \times X$ is a <u>partial order</u> if

$$x \leq x$$
$$x \leq y \text{ and } y \leq z \Rightarrow x \leq z$$
$$x \leq y \text{ and } y \leq x \Rightarrow x = y.$$

<u>Example</u>: Let $A, B, \ldots$ be subsets of a set Z.

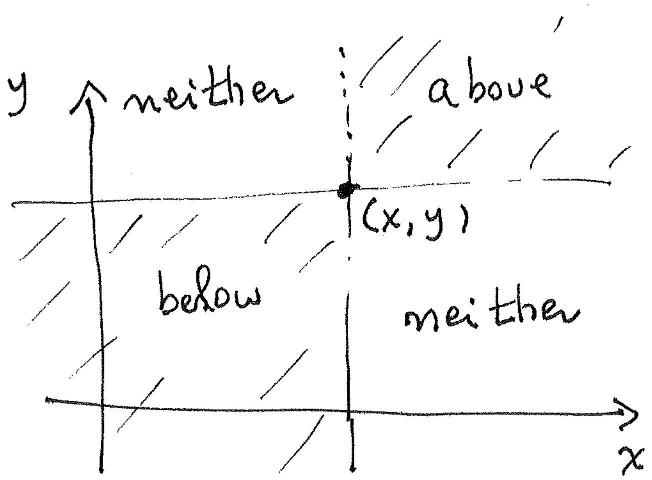
$$A \leq B \equiv A \subset B \quad \text{is a partial order}$$

"ordered by inclusion".

<u>def</u>: A <u>chain</u> is a partial order where

$x \leq y$ or $y \leq x$ for every $x, y \in X$.

Also called a <u>total order</u>.



ex: Let

$(x, y) \leq (x', y')$

iff $x \leq x'$ and $y \leq y'$

Equivalence relations ☆ Very important

def: $E \subset X \times X$ is an equivalence relation if

a) $x E y$ and $y E z \Rightarrow x E z$ "E is transitive"

b) $x E y \Leftrightarrow y E x$ "E is symmetric"

c) $x E x$ "E is reflexive"

for all $x, y, z \in X$.

$[x]_E \equiv \{x' \in X : x' E x\}$ an equivalence class.

$X/E \equiv \{[x]_E : x \in X\}$ the set of equivalence classes.

Thm: X/E covers X without overlapping.

Proof: homework.

$E_q(R) \equiv \{$the intersection of all equivalence relations containing the relation $R\}$

hw: Prove that if E_1 and E_2 are equivalence relations, then so is $E_1 \cap E_2$.

Now we will show that every function has an isomorphism hidden inside it!

Let $X \xrightarrow{f} Y$

Let $x E x'$ iff $f(x) = f(x')$

$$\begin{tikzcd} X \arrow[r, "\text{epi}"] \arrow[rrr, bend left, "f"] & X/E \arrow[r, "\text{iso}"] & \mathrm{Im}\, f \arrow[r, "\text{mono}"] & Y \\ x \arrow[r, mapsto] & {[x]} \arrow[r, mapsto] & f(x) \arrow[r, mapsto] & f(x) \end{tikzcd}$$

This will also work in other categories.

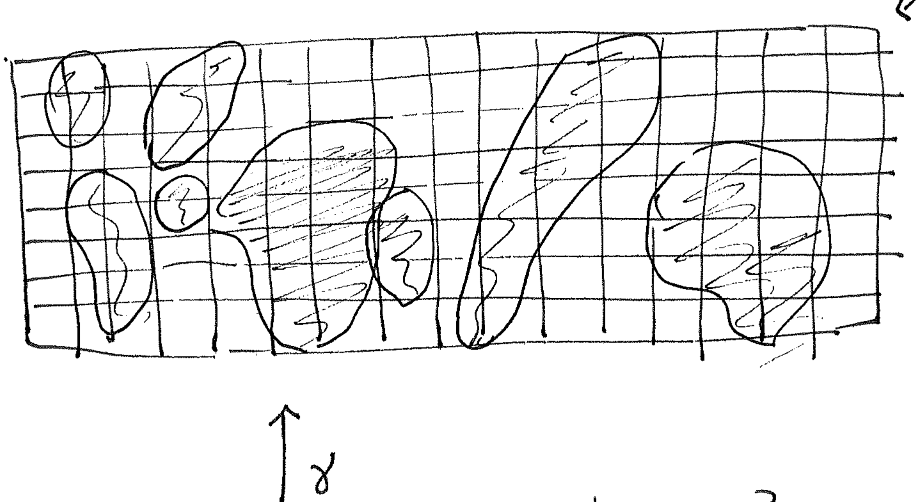
Application:

Cornell e^+e^- synchrotron

$$e^+ + e^- \to \Upsilon' \longrightarrow \gamma + \Upsilon$$

$\Upsilon \hookrightarrow$ hadrons

γ: ~ 100 MeV

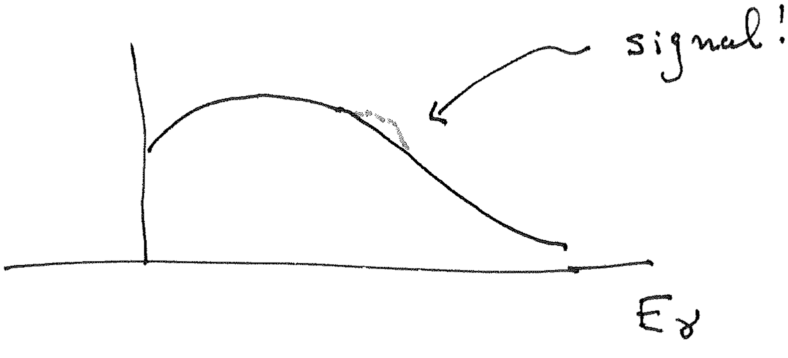


How to find the best clustering?

Ans:

1. Let $x R y$ if y is the "best neighbor of y".
2. Clusters are $X / Eq(R)$.

Algorithm for this is fast: $\mathcal{O}(n \log n)$.



Next time:
Groups Reading:
Geroch Ch. 1-5.