Saul Youssef, 2013

The Cantor-Bernstein theorem

Thm: If $A \overset{f}{\hookrightarrow} B$ and $B \overset{g}{\hookrightarrow} A$, then $A \cong B$.

Proof: By the lemma below, the order preserving function

$$\Psi(S) \equiv g[f[S]^c]^c$$

has a fixed point Z so that

$$Z = g[f[Z]^c]^c \text{ and } Z^c = g[f[Z]^c].$$

Let

$$\alpha : a \mapsto \begin{cases} f(a) & \text{if } a \in Z \\ g^{-1}(a) & \text{if } a \in Z^c. \end{cases}$$

α is 1-1 since g^{-1} is 1-1 where defined. Suppose $b \in B$. If $b \in f[Z]$, then $b = f(a)$ for some $a \in Z$. If $b \in f[Z]^c$, then $a = g(b)$ for some $a \in Z^c \Rightarrow b = g^{-1}(a)$ for some $a \in Z^c$ $\Rightarrow \alpha$ is onto, $\Rightarrow A \cong B$.

Lemma: Given a set X, every order-preserving function $f : \mathcal{P}(X) \to \mathcal{P}(X)$ has a fixed point.

Proof: Let $A \equiv \{ S \in \mathcal{P}(X) : S \leq f(S) \}$ and $a \equiv \bigcup_{S \in A} S$.

Since $f(a) \geq f(s)$ for all $s \in A$, $f(a) \geq a$. On the other hand, $f(a) \leq f(f(a)) \Rightarrow f(a) \in A \Rightarrow f(a) \leq a$.

$\Rightarrow f(a) = a$, f has a fixed point.

Questions

1. Does this work for monomorphisms in other categories?

   – Only sometimes. It works in vector spaces (as we will see), but it does not work for groups.

2. Is $\mathcal{P}(X)$ categorical in some way?

   – Yes! $\mathcal{P}$ is a "functor"

3. Does the lemma have a useful generalization?

   – Yes. It is a special case of the following:

     Thm: A lattice $\mathcal{L}$ is complete iff every order preserving mapping $f: \mathcal{L} \to \mathcal{L}$ has a fixed point.

     See J.B. Nation's book for more.