A tiny bit of Analysis

Riemann integral

Metric spaces, contraction maps

Differentiating under the integral

Picard's theorem for O.D.E.

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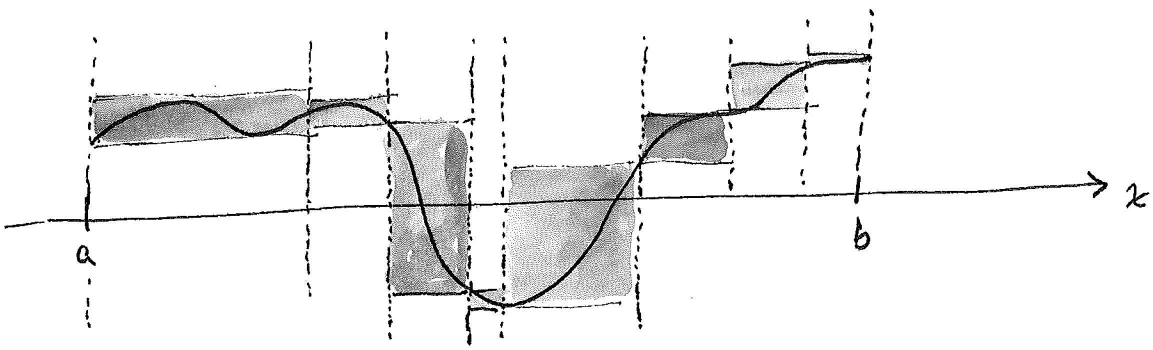
The Riemann Integral

$f : [a,b] \longrightarrow \mathbb{R}$ continuous

Let Π be the directed set of partitions of $[a,b]$, ordered by inclusion. Let

$$\Pi \xrightarrow{K} \mathbb{R} \times \mathbb{R}$$

be the net defined by $K(\pi) \equiv \sum'_{[l,h]\in\pi} (h-l)\cdot f[l,h]$.



Since f is continuous, K is contained in a compact subset of $\mathbb{R} \times \mathbb{R}$. By Geroch THEOREM 35, K has a point of accumulation $[\min, \max] \in \mathbb{R} \times \mathbb{R}$. Since K is also order preserving, $[\min, \max]$ is also the limit of K ($\mathbb{R} \times \mathbb{R}$ ordered by containment). If $\min \neq \max$, a finer partition will make min and max closer. Thus $\min = \max \equiv \int_a^b f(x)\,dx$.

I also want to claim (without proof)

that

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and if $f(x,y)$ is continuous on $\mathbb{R}^2 \to \mathbb{R}$,

then

$$\int_a^b f(x,y)\,dx : \mathbb{R} \to \mathbb{R}$$

is also continuous.

<u>Differentiating under the integral</u> (Used in the Euler-Lagrange equation handout 1.

$$\frac{d}{dy}\int_a^b f(x,y)\,dx \overset{?}{=} \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

Suppose $\frac{\partial f}{\partial y}$ exists. Then

$$F(x,y,\lambda) \equiv \begin{cases} \dfrac{f(x,y+\lambda) - f(x,y)}{\lambda} & \text{if } \lambda \neq 0 \\ \dfrac{\partial f}{\partial y}(x,y) & \text{if } \lambda = 0 \end{cases}$$

is continuous. $\Rightarrow$

$$\int_a^b F(x,y,\lambda)\,dx = \begin{cases} \dfrac{\int_a^b f(x,y+\lambda)\,dx - \int_a^b f(x,y)\,dy}{\lambda} & \text{if } \lambda \neq 0 \\ \int_a^b \dfrac{\partial f}{\partial y}(x,y)\,dx & \text{if } \lambda = 0 \end{cases}$$

is continuous $\Rightarrow$ $\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$ Q.E.D.

Cauchy Sequences in a Metric Space

... are defined just as we did in "Numbers"

<u>def</u>: A sequence $r_1, r_2, r_3, \ldots$ in a metric space is <u>Cauchy</u> if $d(r_i, r_j)$ remains below any chosen $\epsilon > 0$ for all i, j greater than some N.

<u>def</u>: A metric space is <u>complete</u> if every Cauchy sequence converges to a limit.

<u>Factoid</u>: Suppose

$$r_1, r_2, r_3, r_4, \ldots$$

$$\underset{\delta_1}{\longleftrightarrow}\ \underset{\delta_2}{\longleftrightarrow}\ \underset{\delta_3}{\longleftrightarrow}\ \underset{\cdots}{\longleftarrow}$$

where $\delta_{i+1} \leq \delta_i \cdot K$ for some fixed $K < 1$.

Then $r_1, r_2, \ldots$ is a Cauchy sequence.

<u>Proof</u>: $\Delta_{n,m} \leq \delta_n + \delta_{n+1} + \cdots + \delta_{n+m-1}$

$$\leq \delta_n \, (1 + K + K^2 + \ldots + K^{m-1})$$

$$\leq \delta_n \, (1 + K + \ldots) = \delta_n \frac{1}{1-K}.$$

Given any $\epsilon > 0$, choose n such that $\delta_n < \epsilon \cdot (1-K)$.

Contraction Mappings

def: A function $f: X \to X$ on a metric space X is a *contraction map* if $d(f(x), f(y)) \leq d(x,y) \cdot K$ for some fixed $K < 1$.

Thm. A contraction map on a non-empty, complete metric space has a unique fixed point.

Proof: Choose $x \in X$. Then by the previous lemma

$$x,\ f(x),\ f^2(x),\ f^3(x),\ \ldots$$

is a Cauchy sequence which therefore converges to a limit L. Since f is continuous, the net

$$f(x),\ f^2(x),\ f^3(x),\ \ldots$$

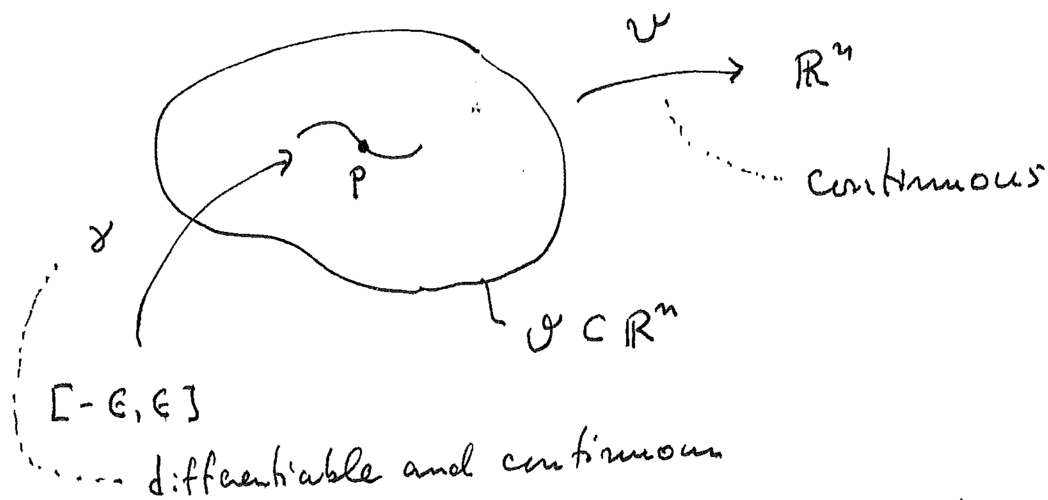
must converge to $f(L)$. $\Rightarrow f(L) = L$ is a fixed point.

Suppose M is another fixed point. Then

$$d(L,M) = d(f(L), f(M)) \leq d(L,M) \cdot K$$

$$\Rightarrow d(L,M) = 0 \Rightarrow L = M.$$

Picard's Theorem



Find $\gamma : [-\epsilon, \epsilon] \to \mathcal{O}$ with $\gamma(0) = p$ and

$$\frac{d\gamma}{dt}(t) = v(\gamma(t)) \quad \text{for } t \in (-\epsilon, \epsilon).$$

$$\Longleftrightarrow \quad \gamma(t) = p + \int_0^t v(\gamma(t))\, dt$$

$$\Longleftrightarrow \quad \Psi(\gamma) \equiv p + \int_0^t v(\gamma(t))\, dt \quad \text{has a fixed point}$$

for $\gamma \in X \equiv \{$ continuous differentiable function from $(-\epsilon, \epsilon)$ to $\mathcal{O}$ $\}$

where $d(\gamma, \gamma') \equiv \| \gamma - \gamma' \|$, $\| \gamma \| \equiv \max\limits_{t \in (-\epsilon, \epsilon)} \| \gamma(t) \|$

is a norm.

We'll try to find conditions such that Ψ is a contraction mapping on X which would guarantee a unique solution.

$$d(\Psi(\gamma), \Psi(\gamma')) = \max_{t \in [-\epsilon, \epsilon]} \left\| \int_0^t v(\gamma(t)) - v(\gamma'(t))\, dt \right\|$$

Suppose that v obeys a "Lipshitz" condition

$$\| v(p) - v(q) \| \leq \| p - q \| \cdot M$$

for some fixed $M > 0$, for all $p, q \in \mathcal{O}$. If this is so, then

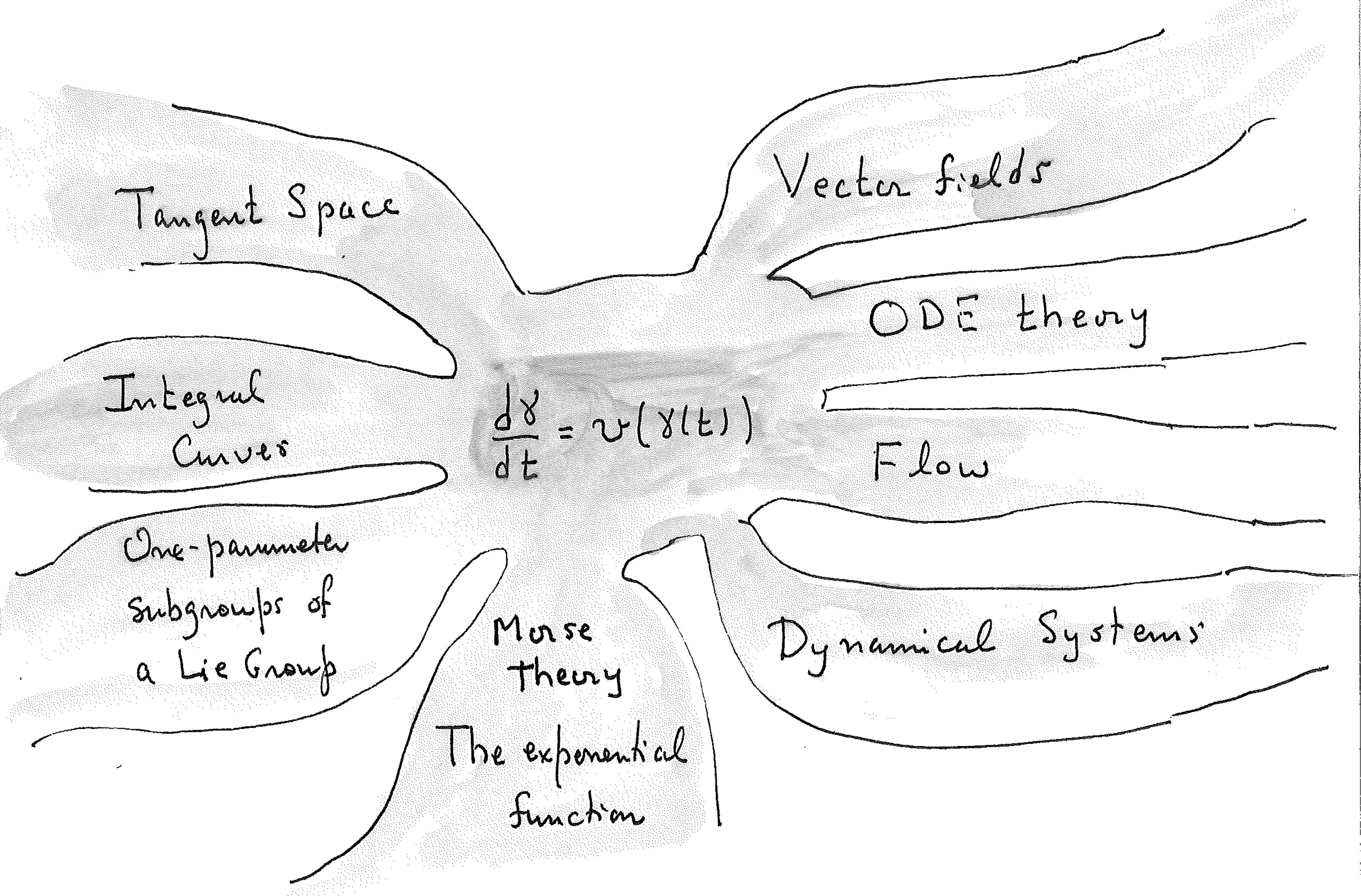
$$d(\Psi(\gamma), \Psi(\gamma')) \leq \epsilon \cdot \max \| \gamma(t) - \gamma'(t) \| \cdot M$$

$$\leq \epsilon \cdot M \cdot d(\gamma, \gamma').$$

$\Rightarrow$ If we choose $\epsilon < 1/M$, Ψ is a contraction and $\dot{\gamma}(t) = v(\gamma(t))$ on $[-\epsilon, \epsilon]$ for a unique function $\gamma : [-\epsilon, \epsilon] \to \mathcal{O}$.

This is a **CROSSROADS**

of many subjects



Subtle behavior is possible even in one dimension, e.g. $v(x) = x^{2/3}$ has two integral curves at $x=0$.