A Reformulation of Quantum Mechanics

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Abstract

We show that the phenomena explained by Quantum Mechanics can alternatively be explained as a breakdown of Probability Theory without the need for wave-particle duality or the idea that a particle does not have a unique path in space. The single particle Lagrangian consistent with the reformulated Quantum Mechanics is derived and specialized to the Schrödinger and Klein–Gordon theories. The usual paradoxes of Quantum Mechanics are explained. A connection to gravity is proposed. Probability Theory is restored in the classical limit.

1. Introduction

Ever since the invention of Quantum Mechanics, there have been problems in the interpretation of the theory. Although predictions of the theory have been uniformly successful, basic questions such as

What is the meaning of the wave-particle duality?

What happens when a wavefunction collapses?

What constitutes a measurement?

What is the role of the observer?

continue to be puzzling. This basic confusion is made flesh by the various "paradoxes" such as the Einstein–Podolsky–Rosen(EPR) paradox[1] and the paradox of Schrödinger's cat.[2] There have been many attempts to answer these questions by either changing Quantum Mechanics or by changing it's interpretation.[3] These include Hidden Variable theories,[4] Semi–classical theories,[5] Collapse Interpretations,[6] the Many Worlds Interpretation,[7] Advanced Action Interpretations,[8] Non-Local Models,[9] the Consistent Histories Model[10] and Quantum Logics.[11] Here, rather than attempting to work within the framework of standard Quantum Mechanics, a reformulated theory is proposed which is shown to give the same predictions as the standard Quantum Theory and is shown to give satisfying answers to the paradoxes.

2. The Two Slit Experiment

Quantum Mechanics is sometimes motivated by the well known two slit experiment. In this situation, a source emits particles of a particular energy and type, one at a time, towards a barrier with two slits. Behind the barrier is a screen and a detector which can be moved in the transverse (x) direction and which can be used to count the number of particles which arrive at each x position. With either of the slits blocked, the frequency profile in x is smooth with one peak. To predict what happens when both slits are open, consider the following proposition:

Proposition A: Each particle *either* goes through slit #1 *or* it goes through slit #2.

If Proposition A is true, then the frequency profile with both slits open must be the sum of the profiles with one or the other of the slits blocked. Since the actual result is an interference pattern with many peaks, Proposition A must be false.

This analysis of the two slit experiment leads to the conclusion that particles have a "wave nature" and that each individual particle goes through both slits at once. However, there is an alternative interpretation of the experiment where Proposition A is accepted as *true* and the interference result is explained as a breakdown of Probability Theory itself, rather than due to new properties of the particle. A failure of Probability Theory could result in

$$P(x) \neq P(x \textbf{ and } slit\#1) + P(x \textbf{ and } slit\#2)$$

which would avoid a contradiction. Thus, rather then introducing new mechanics as such, we explore the alternative of modifying Probability Theory.

3. Axioms and Definitions

Probability Theory can be axiomized as follows. Consider a set of propositions $\mathcal{P}$ closed under **and**($\wedge$), **or**($\vee$) and **not**($\neg$). Thus, $\mathcal{P}$ is a set of questions with definite yes or no answers. A **probability** is a function mapping $\mathcal{P} \times \mathcal{P}$ into the interval $[0, 1]$ satisfying

$$\begin{aligned}
&P(b \wedge c|a) = P(b|a)P(c|a \wedge b), \\
&P(b|a) + P(\neg b|a) = 1, \\
&P(\neg a|a) = 0
\end{aligned}$$

for all $a, b, c \in \mathcal{P}$, where "$P(b|a)$" is the probability that b is true given a, etc. These axioms are highly constrained by the following, even more primitive assumptions:

$P(b \wedge c|a) = F(P(b|a), P(c|a \wedge b))$ for some fixed function F,
$P(\neg b|a) = G(P(b|a))$ for some fixed function G.

With these assumptions, R.T. Cox has shown that the structure of $\mathcal{P}$ leads, almost unambiguously, to the standard axioms.[12] The simplest way to introduce complex numbers into the theory while maintaining the primitive assumptions of Cox is to replace the interval $[0, 1]$ with the set of complex numbers ($\mathbf{C}$). To distinguish the modified Probability Theory from the original, introduce an arrow "$\rightarrow$" to signify a binary function mapping $\mathcal{P} \times \mathcal{P}$ into $\mathbf{C}$. The new complex probability satisfies

$$\begin{aligned}
&(a \rightarrow b \wedge c) = (a \rightarrow b) \times (a \wedge b \rightarrow c) \\
&(a \rightarrow b) + (a \rightarrow \neg b) = 1 \\
&(a \rightarrow \neg a) = 0
\end{aligned}$$

for all $a, b, c \in \mathcal{P}$. We take this modified probability theory to be the reformulated Quantum Mechanics. To avoid confusion, the term "probability" will refer to a standard probability and the term "amplitude" will be used for the complex probability.

Many of the familiar results of Probability Theory follow easily from the new axioms. For example, for all $a, b, c \in \mathcal{P}$,

a) $a \to a = 1$
b) $a \to T = 1$
c) $a \to F = 0$
d) $(a \to b \vee c) = (a \to b) + (a \to c) - (a \to b \wedge c)$
e) If $(a \to b) \neq 0$, then $(a \wedge b \to c) = (a \to c)(a \wedge c \to b)/(a \to b)$ (Bayes Theorem)

where "T" is the question whose answer is always *true* and where "F" is the question whose answer is always *false*. Note that the new "amplitude" is not the same as the amplitude in standard Quantum Mechanics. For example, it is not generally true that $(a \to b) = (b \to a)^*$, and $|a \to b|^2$ is not generally a probability.

In order to make a dynamical description of a given system, choose $\mathcal{P}$ to contain questions ordered with a parameter $t \in [0, T]$. Such questions are denoted "a, t" meaning "a is true at time t." Since $\mathcal{P}$ is closed under the standard boolean operations, questions such as $a, t \wedge b, t'$ are also included, even if $t \neq t'$. In addition to a global choice of questions about a system, it is useful to define what it means for a set of questions to form a complete specification. A set of questions $\mathcal{P}_t = \{x, t : x \in U, t \in [0, T]\} \subset \mathcal{P}$ is **complete** if, for all $x, t \in \mathcal{P}_t$, $x, t' \in \mathcal{P}_t$, $y, t \in \mathcal{P}_t$, $a, t \in \mathcal{P}$ and $b, t'' \in \mathcal{P}$:

a) $x \neq y \Rightarrow x, t \wedge y, t = F$,
b) $(a, t \to b, t'') = (a, t \to (\bigvee_{x \in U} x, t') \wedge b, t'')$ if $t' \in (t, t'')$,
c) $(a, t \wedge x, t' \to b, t'') = (x, t' \to b, t'')$ if $t' \in (t, t'')$.

As a concrete example of a complete set of questions, let $\mathbf{Q} = \{x, t : x \in \mathbb{R}^3, t \in [0, T]\}$ be the set of questions about a single scalar particle in three dimensions. Informally, $\mathbf{Q}$ is complete if (a) the particle cannot have two positions at once, (b) if the particle moves from time t to time t'', it must have a position $x \in \mathbb{R}^3$ at any intermediate time t' and (c) the future amplitudes of a particle are determined by it's last known position and the past amplitudes of a particle are determined by it's earliest known position.

4. Single Particle Dynamics

Consider a single particle in d-dimensions and assume that

1) The global set of propositions $\mathcal{P}$ includes all questions about the position of the particle in $\mathbb{R}^d$ specified at a particular "time" $t \in [0, T]$,
2) $\mathcal{P}_t = \{x, t : x \in \mathbb{R}^d, t \in [0, T]\} \subset \mathcal{P}$ is complete.

We wish to calculate $(a, t_0 \to x, t)$ where a is some prior statement about the system known to be true at time $t_0 < t$. The desired amplitude can be expanded using the axioms and completeness into

$$(a, t_0 \to x, t) = (a, t_0 \to (\bigvee_{x_1 \in \mathbb{R}^d} x_1, t_1) \wedge x, t)$$

$$
\begin{aligned}
&= \textstyle\sum_{x_1} (a, t_0 \to x_1, t_1 \wedge x, t) \\
&= \textstyle\sum_{x_1} (a, t_0 \to x_1, t_1)(a, t_0 \wedge x_1, t_1 \to x, t) \\
&= \textstyle\sum_{x_1} (a, t_0 \to x_1, t_1)(x_1, t_1 \to x, t) \\
&= \textstyle\sum_{x_1} \sum_{x_2} \ldots \sum_{x_{n-1}} (a, t_0 \to x_1, t_1)(x_1, t_1 \to x_2, t_2) \ldots (x_{n-1}, t_{n-1} \to x_n, t_n)
\end{aligned}
$$

where $x_1, x_2, \ldots x_{n-1}$ are summed over $\mathbb{R}^d$, with $x_n = x$, $t_n = t$ and where $t_{j+1} - t_j = \tau$ for $j = 0, 1, ..., n-1$. Thus, $(a, t_0 \to x, t)$ is given by a path integral (see Refs. 13-15) of products of the form

$$\phi(x, z, \tau) = (x, t \to x + z, t + \tau)$$

where we have assumed that ϕ is independent of absolute time. Let each time interval τ be further divided into N sub-intervals with $\epsilon = \tau/N$ and

$$\mu(x, z, \epsilon) = (x, t \to x + z, t + \epsilon)$$

where μ is the "microscopic amplitude." As with the path integral above, ϕ can be expanded into N convolutions of μ. Assume that τ can be chosen small enough that $\mu(x, z, \epsilon)$ can be taken to be independent of x within each term of the original path integral. Then

$$\widetilde{\phi}(x, u, \tau) = \widetilde{\mu}(x, u, \epsilon)^N$$

where[16] $\widetilde{\mu}(x, u, \epsilon) = \int d^d z \; \mu(x, z, \epsilon) e^{i u_j z_j}$ and $\widetilde{\phi}(x, u, \tau) = \int d^d z \; \phi(x, z, \tau) e^{i u_j z_j}$. After expanding μ: $\mu(x, z, \epsilon) = \delta(z) + \epsilon \mu_\epsilon(x, z, 0) + O(\epsilon^2)$ and introducing the moments

$$\nu_0(x) = \int d^d z \; \mu_\epsilon(x, z, 0),$$

$$\nu_j(x) = \int d^d z \; \mu_\epsilon(x, z, 0) z_j,$$

$$\nu_{jk}(x) = \int d^d z \; \mu_\epsilon(x, z, 0) z_j z_k,$$

the $\epsilon \to 0$ limit of ϕ can be written as

$$\phi(x, z, \tau) = \frac{e^{\tau \nu_0}}{(2\pi)^d} \int d^d u \; \exp(-i u_j z_j + \tau(i u_j \nu_j - \frac{1}{2} u_j u_k \nu_{jk})).$$

If ν_{jk} can be diagonalized, a complex orthogonal matrix M can be found[17] such that $M^T_{lj} \nu_{jk} M_{km} = \delta_{lm}/\omega_l$. Letting $W_{jk} = M_{jl} M^T_{lk} \omega_l$ and $\sigma(x) = \det[M(x)] \in \{+1, -1\}$, then

$$\phi(x, z, \tau) = \frac{\sigma}{(2\pi\tau)^{\frac{d}{2}} \sqrt{\det[\nu]}} \exp(-\frac{1}{2\tau}(z_j - \tau\nu_j) W_{jk} (z_k - \tau\nu_k) + \tau\nu_0)$$

is the general form of the propagator. Equating the exponent of the propagator with i times the action over an interval τ and assuming that σ and $\det[\nu]$ are constant, the most general single particle time independent Lagrangian consistent with quantum mechanics is

$$\mathcal{L}(x,v) = \frac{i}{2}(v_j - \nu_j)W_{jk}(v_k - \nu_k) - i\nu_0$$

where $v_j = z_j/\tau$ is the velocity and where $\nu_0(x)$, $\nu_j(x)$ and $W_{jk}(x)$ are moments of the microscopic amplitude.

If $d = 3$ and $W_{jk} = \delta_{jk}\omega(x)$, then $\det[\nu] = \omega(x)^{-d}$ and the definitions

$$\omega(x) = -im,$$

$$\nu_0(x) = -ig\phi_0(x),$$
$$\nu_j(x) = \frac{g}{m}A_j(x),$$

result in the Lagrangian for the Schrödinger equation with a particle of mass m, charge g, with a vector potential $A_j(x)$ and a scalar potential $\phi_0(x)$. Thus, any microscopic amplitude consistent with real scalar and vector potentials and a constant mass leads to the Schrödinger equation where the mass, vector and scalar potentials appear as moments. Notice that all the gauge invariant terms are present.

Given any mass, vector and scalar potentials, there is a microscopic amplitude such that $(a, t_0 \to x, t)$ is a solution of the Schrödinger equation with initial condition $\psi_0(x) = (a, t_0 \to x, t_0)$. Conversely, a given solution $\psi(x,t)$ of the Schrödinger equation can be made into an amplitude by normalizing ψ so that $\int_0^T dt \int d^d x \ \psi(x,t) = 1$. Any difficulties caused by wavefunctions with $\int d^d x \ \psi(x,t) = 0$ can be avoided either by making a slight change in the initial conditions or by normalizing ψ in a finite volume.

As an example and as a test of consistency, Bayes Theorem can be used to add new information to an existing wavefunction. The resulting amplitude must also be a solution of Schrödinger's equation. Given that a particle is at $x \in \mathbb{R}^d$ at time t_0 and that the particle is in a volume $R_y \subset \mathbb{R}^d$ at time $t_0 + \Delta$, we wish to compute the amplitude for the particle to be at a position $z \in \mathbb{R}^d$ at a later time $t_0 + 2\Delta$. To simplify notation, suppress the time specifications and denote the statement that the particle is in R_y at the intermediate time by "in R_y." Using Bayes theorem,

$$(x \wedge \text{in } R_y \to z) = (x \to z)\frac{(x \wedge z \to \text{in } R_y)}{(x \to \text{in } R_y)}$$

and since

$$(x \wedge z \to \text{in } R_y) = (x \wedge z \to \bigvee_{y \in \mathbb{R}^d} (y \wedge y \in R_y)) = \sum_{y \in R_y} (x \wedge z \to y)$$

and since Bayes theorem and completeness implies that

$$(x \wedge z \to y) = (x \to y)\frac{(x \wedge y \to z)}{(x \to z)} = (x \to y)\frac{(y \to z)}{(x \to z)}$$

the desired amplitude can be written

$$(x \wedge \text{in } R_y \to z) = \frac{1}{(x \to \text{in } R_y)} \sum_{y \in R_y} (x \to y)(y \to z)$$

provided that $(x \to \text{in } R_y)$ and $(x \to z)$ are nonzero. Thus, since $(y \to z)$ is a solution to the Schrödinger equation, then so is $(x \wedge \text{in } R_y \to z)$. Similarly, the amplitude to arrive at z and to be in R_y at the intermediate time is given by

$$(x \to \text{in } R_y \wedge z) = \sum_{y \in R_y} (x \to y)(y \to z)$$

which is also a solution to the Schrödinger equation.

The above considerations for the Schrödinger equation also apply to the four dimensional case with t identified as a path length parameter rather than the fourth component of x. The choice of moments $\omega(x) = 1$, $\nu_0(x) = -\frac{1}{2}(g^2 A_j(x)A_j(x) + m^2)$, and $\nu_j(x) = igA_j(x)$ results in the Lagrangian for Klein-Gordon equation if, as usual, the path length parameter is integrated out. Notice that without the arbitrary choice of $\nu_0(x)$ to cancel the photon mass term, the theory would violate gauge invariance.

5. Restoration of Probability Theory

The restoration of standard Probability Theory can be illustrated by taking the classical limit of the amplitude for a particle to be found in a volume $V \subset \mathbb{R}^d$ at some time $t > t_0$. Given that i is known at time t_0, the amplitude to find the particle in a volume $V \subset \mathbb{R}^d$ at time t is given by the marginal amplitude

$$(i, t_0 \wedge \bigvee_{x \in \mathbb{R}^d} x, t \to \text{in } V, t) = \sum_{x \in V} (i, t_0 \to x, t) / \sum_{x \in \mathbb{R}^d} (i, t_0 \to x, t).$$

If, in the classical limit of Quantum Mechanics, a particle can be described by a wave packet which follows the classical trajectory[18] and if the volume V is much larger than the size of the wave packet, then the amplitude to find the particle in V at time t is 1 if it contains the classical trajectory and zero otherwise. If the initial knowledge of a classical particle can be described by a probability distribution in position–momentum phase space, this situation can be represented as a sum of wave packets with positive weights. The amplitude to be in a region V is then just the sum of the weights for the wave packets which are contained by V at time t. Thus, in this limit, the amplitude has been reduced to a standard positive probability.

In some Quantum theories, a conserved current can be constructed from the wavefunction with a positive current density. The density $|\psi(x,t)|^2$ in the original Schrödinger theory is an example of this. Outside the classical limit, the probability meaning of the amplitude does not interfere with a probability interpretation of $|\psi(x,t)|^2$ since negative or complex probabilities cannot have a frequency interpretation. In the classical limit as described above, either the amplitude or the magnitude of the wavefunction squared can be interpreted as a probability density. Since, in the latter case, the initial weights would have been replaced by their square roots, and since the initial wave packets remain separated, the two interpretations give identical results.

6. The Meaning of the Microscopic Amplitude

In general, extracting dynamics from this theory can proceed from a choice of a complete set of questions. Given a complete set of questions $\mathcal{P}_t$ describing a system at "time" t, any desired amplitude can be expanded into a path integral. Repeated application of a microscopic amplitude then results in a propagator and an effective Lagrangian where masses and background fields emerge as moments. This emergence of an effective Lagrangian depending only on the lowest moments is just a version of the Central Limit Theorem. Simplifying assumptions on the form of the microscopic amplitude may also simplify the form of the Lagrangian. The formulation of the theory does not depend on the metric structure of $\mathbb{R}^d$ and could be formulated on a suitably smooth manifold or on a discrete set.

In both the Schrödinger and Klein-Gordon theories, constraints have been imposed on the microscopic amplitude in order to avoid unwanted terms in the effective Lagrangian. Since there is no other mechanism for keeping $W_{jk}(x)$ constant, it is tempting to attribute this feature to the flatness of space-time and to equate $W_{jk}(x)$ with the space-time metric $g_{\mu\nu}(x)$ from General Relativity.[19] This would then result in a quantum theory of a scalar particle in a static gravitational field. In any case, since $\phi(x, z, \tau)$ is the most general single particle time independent propagator consistent with Quantum Mechanics, ϕ must also describe a scalar particle in a static gravitational field, assuming that such a theory exists.

7. Multi-Particle Systems

The Quantum Mechanics of multi-particle systems can be described by choosing new complete sets of questions. In the case of two distinguishable scalar particles in $\mathbb{R}^d$, choose

$$\mathcal{P}_t = \{x_1, x_2, t : x_1, x_2 \in \mathbb{R}^d\}$$

where "x_1, x_2, t" is shorthand notation for "particle 1 is at x_1 and particle 2 is at x_2 at time t." On the other hand, if the particles are identical, this must be reflected in a different set of questions needed to describe the system:

$$\mathcal{P}_t = \{\{x_1, x_2\}, t : x_1, x_2 \in \mathbb{R}^d\}$$

where "$\{x_1, x_2\}, t$" is short for "the positions of the two particles are $\{x_1, x_2\}$ at time t." It is easy to verify that the case of n distinguishable particles in d-dimensions, with

$$\mathcal{P}_t = \{x, t : x \in (\mathbb{R}^d)^n\},$$

is isomorphic to a single particle in $n \times d$ dimensions, so the results of section 4 can be used in this case. From this point of view, a natural starting point for scalar field theory is the set of questions $\mathcal{P}_t = \{\varphi, t : \varphi \text{ continuous}, \varphi : \mathbb{R}^d \to \mathbb{R}\}$.

8. The Bayesian View of Amplitudes

With the results of the last sections, it is clear that the reformulated version of Quantum Mechanics has the predictive power of ordinary Quantum Mechanics, at least for a scalar particle. With the assumption that this is correct, then, we are in a position to understand the well known "paradoxes" of Quantum Mechanics such as the Einstein–Podolsky–Rosen paradox and the meaning of the collapse of the wavefunction.

The key to understanding these problems is to realize the status of the amplitudes with respect to the real physical system. Just as in the Bayesian view of probabilities,[20] our amplitudes do not constitute the "state of the system," they only constitute the best estimate of various truths given some prior information. An amplitude such as $(a \to b)$ is the best estimate of the truth of b given that a is known. All of the mysterious aspects of the Quantum paradoxes are due to mistaking the wavefunction for the state of the system. Unfortunately, this mistake is enforced by the language of standard Quantum Mechanics e.g. a vector in a Hilbert space is called a "state vector."

It is easy to see how the Bayesian viewpoint resolves the usual paradoxes, since the "collapse of the wavefunction" then has no more significance than the "collapse of the probability distribution" when a die is thrown. The difficulties in analyzing the EPR experiment are only due to the fact that the two observers have different information about the system and describe the system by two different wavefunctions. This "knowledge" explanation of the paradoxes has been suggested before[20,21] in the context of standard Quantum Mechanics. Here we add a demonstration that, under suitable conditions, our "knowledge of the system" obeys Schrödinger's equation.

Armed with the reformulated Quantum Mechanics and with the Bayesian interpretation of amplitudes, reconsider the questions raised in the introduction:

What is the meaning of the wave-particle duality?
There is no need for a wave-particle duality. The apparent wave nature of particles can be explained by a breakdown of Probability Theory.

What happens when a wavefunction collapses?
The collapse of a wavefunction is not something which happens within the system. It is simply a change in the description of the system which is made when new information becomes apparent.

What constitutes a measurement?
A physical process is called a "measurement" when information from that process has been used as known information in estimating amplitudes. In all other respects, a measurement is a physical process like any other.

What is the role of the observer?
There is no particular role for an observer in the theory besides the person who decides what information to use in constructing the wavefunction.

In addition to resolving paradoxes, the Bayesian interpretation of amplitudes may have an importance in practical applications. With the realization that the wavefunction represents only our knowledge comes the freedom to make use of prior information to systematically improve the amplitudes using the complex version of Bayes Theorem. The analogous program in Probability Theory has had spectacular practical successes.[20]

It may be important to point out that the results so far do not depend on the existence of any random phenomenon. Cox's result shows that the axioms of either Probability Theory or Quantum Mechanics are just a consistent calculus for reasoning about propositions, whether or not those propositions are questions about a random variable and whether or not the question corresponds to a frequency. Thus, even though we are describing Quantum Mechanics by a modified Probability Theory, this does not imply that there is any random phenomenon associated with Quantum Mechanics.

9. Summary and Further Developments

We have shown that the effects of Quantum Mechanics can be explained as a failure of standard Probability Theory without assuming that a particle has a "wave nature" and travels by more than one simultaneous path, as in the two slit experiment. Even though we have assumed that a unique path for a particle exists, this does not imply that we can determine what that path is. In fact, it remains true that in the two slit experiment, one cannot determine which slit a particle has gone through without destroying the interference pattern since this analysis follows from the propagator corresponding to the Schrödinger equation.[14] In addition to the considerations in section 2, Bell's inequalities can be quoted in support of the view that Probability Theory must be altered since, otherwise, experimental evidence forces the conclusion that Quantum Mechanics violates locality.[23]

In order to accommodate Quantum phenomena, Probability Theory is modified by simply by replacing the interval $[0, 1]$ with $\mathbf{C}$ in the Bayesian axioms. This complex Probability Theory is consistent with the primitive assumptions and analysis of Cox and is proposed as a reformulation of Quantum Mechanics. The view that the amplitudes represent knowledge rather than the state of the system is used to explain the standard paradoxes associated with the collapse of the wavefunction and the role of the observer in the theory. The realization that amplitudes represent knowledge only also suggests the use of the complex Bayes theorem to improve calculations of amplitudes just as the ordinary Bayes theorem is used to improve probability distributions in Bayesian Inference. Although Quantum Mechanics is proposed as a Probability Theory, our analysis does not require any randomness.

Given the basic theory, dynamics for a single scalar particle in d-dimensions is determined by choosing a "complete set of questions" parameterized by a time-like parameter t. An arbitrary choice for the "microscopic amplitude" $(x, t \to x + z, t + \epsilon)$ then results in an effective Lagrangian where the particle mass and the background fields emerge as moments. We have found the most general single particle time independent Lagrangian consistent with the reformulated Quantum Mechanics. In the case of $d = 3$, the Schrödinger equation is derived, including all of the gauge invariant interaction terms. Similarly, $d = 4$ leads to the Klein-Gordon equation. In the case of the Klein-Gordon equation, a gauge violating photon mass term may be present, depending on the choice of $\nu_0(x)$. Other than that, only gauge invariant terms appear in the effective Lagrangian. In the classical limit, the amplitudes can be chosen to be real and positive which implies the restoration of standard Probability Theory. Multi-particle dynamics is discussed and the problem of n distinguishable particles in d dimensions is identified with the solution for a single scalar particle in $n \times d$ dimensions. In order to explain constraints on the moments $W_{jk}(x)$, we have suggested identifying these moments with the metric $g_{\mu\nu}(x)$ from General Relativity.

Although development of the theory so far has proceeded without problems, there are many avenues of investigation which remain open including spin systems, the Dirac equation, development of multi-particle systems, Field Theory and the significance of any connection to Gravity. An understanding is needed of the relationship of gauge invariance to the constraints implied by Quantum Mechanics alone. The complex version of Bayes theorem should have practical applications.

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