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Quantum phenomena via complex measure: Holomorphic extension

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The complex measure theoretic approach proposed earlier is reviewed and a general version of density matrix as well as conditional density matrix is introduced. The holomorphic extension of the complex measure density (CMD) is identified to be the Wigner distribution function of the conventional quantum mechanical theory. A variety of situations in quantum optical phenomena are discussed within such a holomorphic complex measure theoretic framework. A model of a quantum oscillator in interaction with a bath is analyzed and explicit solution for the CMD of the coordinate as well as the Wigner distribution function is obtained. A brief discussion on the assignment of probability to path history of the test oscillator is provided.

1 Introduction

The object of this contribution is to show that the complex measure density studied in earlier contributions [22,23,27] admits of a special holomorphic extension that leads interestingly to a general version of density matrix and Wigner distribution function. To demonstrate the power of the technique we show how explicit expressions can be derived in a variety of situations in quantum optical phenomena. We also show how the Caldeira -Legget model can be elegantly handled thus paving the way for answering many interesting questions related to the past history in quantum phenomena.

In earlier contributions [22–25, 27, 28] complex measurable processes and their extensions were studied with the specific objective of describing various facets of quantum phenomena. The sole motivation for the search for an alternative structure is to examine the possibility of a framework that incorporates nondeterminism right from the beginning and is also possibly divergence free. The very complex nature of the measure structure built into the system is sufficient to accommodate interference and ensure violation of Bell's inequalities. Of course the violation of Bell's inequalities had been dealt with earlier from many angles; however the work most pertinent to our approach is that of Sudarshan and Rothman [30] wherein it is shown that the relaxation of the constraint of positive definiteness for probabilities ensures violation of Bell's inequalities. In a parallel contribution Yousseff [31–33] has advocated the use of 'exotic' probability theory wherein conditional complex probabilities are introduced thus accommodating violation of Bell's inequalities in a natural manner. The idea that a modified version of probability theory may be the appropriate tool to arrive at a theory of quantum phenomena is not entirely new and perhaps goes back to Dirac (see [34]). Our approach based on complex measures/measurable processes and their extension is direct and simple in the sense that it accommodates interference and internal motion ensuring at the same time (consistency with) the violation of Bell's inequalities. Thus a frame work containing these basic ingredients called the complex/extended complex (quaternion)measure theoretic framework (CMTF/QMTF) is used

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for the study of the harmonic oscillator of the free as well as forced type that has yielded interesting and useful results for the general formulation of quantum phenomena (Srinivasan [22, 23] referred to as I and II hereafter). Since the electromagnetic field can be viewed as a linear sum of harmonic oscillators, the analysis of the forced harmonic oscillator leads, just as in path integral theory, to an elegant method of handling Lamb shift; however unlike path integral approach, it turns out that there are no divergences and hence no need for the ultra violet cut off, the estimate of the non-relativistic Lamb shift being very close to that obtained by using the renormalization theory and Feynman cut off. The general analysis presented in II essentially makes full use of the fact that the square root of the complex measure density (CMD) being square integrable provides a natural L_2-setting; thus a Hilbert space can be constructed and the properties of many of the CMD's can be established elegantly. In this connection it is worth mentioning that while the analysis by the construction of the Hilbert space generated by the CMD's rather the square roots enables us to perform with ease the necessary computations, the basic frame work is essentially a complex/extended measure space. However in another context when we deal with the coherent state and its ramifications (Srinivasan [24] referred to as III), the connection to the Hilbert space operator approach is direct and in some instances the results flowing from CMTF may appear identical with the conventional operator theory despite the fact that CMTF does not contain the basic ingredients of Fock space. In view of these developments it is considered worth while to examine whether the notion of density matrix in a more general form can be constructed and more particularly whether a holomorphic extension of the CMD can describe various facets of quantum phenomena in relation to the coherent state.

At this juncture it is worth mentioning that the idea that the complex probability or parameters can be useful in the description of physical phenomena has a long history. Quite early in the last century Fürth [10] noted the formal analogy between a real Brownian motion and the Schrödinger equation by using a complex diffusion coefficient. In an entirely different context, Cox [4] introduced complex probabilities in the classical theory of stochastic processes; he was primarily concerned with an interpretation of Erlang distribution in the study of renewal processes. Lavenda and Santamato [17,18] following Fürth introduced complex measurable processes using Ito's concept of generalized uniform complex measures [14, 15]. Ito rather provided a mathematical structure to the Feynman path integrals and also had shown how complex measurable processes arising therefrom can be interpreted in a generalized sense thus providing a justification for the interpretation of Feynman path integrals particularly by Gelfand and Yagolam [11]. Lavenda and Santamato [18] essentially made use of these results to interpret the complex measures to arrive at the propagating kernel and make connection to the formula obtained by the use of Feynman path integral. In recent times Youssef [31–33] has used complex probabilities for reformulation of quantum mechanical postulates and advocated the use of exotic probability for the description of quantum phenomena.

The layout of the paper is as follows. In Sect. 2 we make a quick survey of the basic features of CMTF and show how the notion of the density matrix can be introduced in CMTF in a rather more general form. In particular the conditional density matrix is used thus providing a useful generalization. Then in Sect. 3 we deal with the harmonic oscillator and demonstrate how the holomorphic complex measure density reduces to the Wigner distribution. The coherent state is given a new representation in the holomorphic framework and the density matrix is introduced in a natural way with its characteristic property of the diagonal element coinciding with Wigner distribution. Then we deal with a number of interesting situations in quantum optical phenomena and demonstrate how computations can be performed with great ease. Then in Sect. 4 we show how the Caldeira-Legget model of a test oscillator in interaction with a bath can be handled elegantly. The final section contains a summary and a discussion of the various results obtained.

2 Complex measure density and its properties

We now make a short review of the basic features of the complex measure theoretic framework (CMTF). The starting point is a measurable space $(\Omega, \mathbf{B})$; if μ_1 and μ_2 are any two signed measures defined over $(\Omega, \mathbf{B})$, the complex measure λ is defined by $\lambda = \mu_1 + \mathrm{i}\mu_2$. We introduce random variables and stochastic processes in the same manner as in the case of standard probability theory. A detailed account is provided in an earlier contribution [27]. In the case of complex measurable processes we impose the constraint $\lambda(\Omega) = 1$ to ensure that in the case of Markov processes Chapman Kolmogorov relation can be expressed as a differential equation under further appropriate constraints. Further we impose $|\lambda(A)| < \infty$ for any complex measurable set $A \in \mathbf{B}$. Contact with physics of quantum phenomena by regarding the coordinate as a complex measurable process. Thus we can denote by $\{X(t)\}$ the complex measurable process in question and impose the Markov constraint on it; the process is then characterized by its transition complex measure density $f_2(x, t|x_0, t_0)$ which is the CMD of $X(t)$ conditional on $X(t_0) = x_0$. If we further assume that the moment functions of the transition CMD defined by

$$a_n(y, t, \Delta) = \int (x - y)^n f_2(x, t + \Delta|y, t)\mathrm{d}x \tag{2.1}$$

satisfy

$$\lim_{\Delta \to 0} a_1(y, t, \Delta)/\Delta = a(y) \ , \tag{2.2}$$

$$\lim_{\Delta \to 0} a_2(y, t, \Delta)/\Delta = b(y) \ , \tag{2.3}$$

$$\lim_{\Delta \to 0} a_n(y, t, \Delta)/\Delta = 0 \ , \qquad n > 2 \ , \tag{2.4}$$

then f_2 satisfies the Fokker Planck equation

$$\frac{\partial f_2(x, t|x_0, t_0)}{\partial t} = -\frac{\partial}{\partial x}\left[a(x) f_2(x, t|x_0, t_0)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[b(x) f_2(x, t|x_0, t_0)\right]. \tag{2.5}$$

The above equation is easily extended to cover the case when $\{X(t)\}$ is a vector process. The solution of (2.5) yields in general the CMD; to relate the CMD to actual physical situation at ground reality when experimental results relate to frequency ratios, we need to make a measure transformation. There are two measures that are positive definite as discussed earlier [27]; the first known as the modulus measure is defined for any set $A \in \mathbf{B}$ by

$$|\lambda|(A) = \sup |\int_A f \mathrm{d}\lambda|, \tag{2.6}$$

where the supremum is extended over f for all functions f such that $|f(x)| \leq 1$. It had been shown in II that from CMTF point of view this is the appropriate measure that is consistent with the Born interpretation of the conventional operator theory. The second is the so called modulus square measure and can be introduced in an analogous way by starting from the modulus measure. This measure can be used in the case of quasi-stationary states where the total measure is zero, a property which enables us to define measures for the individual states essentially in a relative sense. Next we note that the square root of the CMD is square integrable over $(-\infty, \infty)$ by virtue of the constraint $|\lambda(A)| < \infty$ and hence we have a natural L_2-setting; in fact a Hilbert space can be constructed thus paving the way for an efficient analysis that can lead to tangible results.

Next we consider some specific CMD's that correspond to quantum harmonic oscillators; there are two choices of $a(x)$ and $b(x)$ that cover a wide variety of quantum phenomena:

$$\text{(i)} \quad a(x) = -\mathrm{i}\omega x \ , \tag{2.7}$$

$$\text{(ii)} \quad a(x) = -\mathrm{i}\omega(x - \alpha), \quad \alpha \text{ a complex parameter.} \tag{2.8}$$

We take $b(x)$ to be a constant equal to $\mathrm{i}\hbar/m$; the primary motivation for this choice is to make a very early contact with physics of quantum phenomena. The detailed analysis leading to the solution of (2.5) for the choice (2.7) is provided in paper I; it is sufficient to note that in this case f_2 is a function of $t - t_0$ only. Setting $t_0 = 0$ and defining $\pi(x, t|x_0)$ by

$$\pi(x, t|x_0) = f_2(x, t|x_0, t_0) \tag{2.9}$$

the final solution is explicitly given by

$$\pi(x, t|x_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-(x - x_0\mathrm{e}^{-\mathrm{i}\omega t})^2/2\sigma^2\right\} \tag{2.10}$$

where

$$\sigma^2 = (\hbar/2m\omega)\,(1 - \mathrm{e}^{-2\mathrm{i}\omega t}). \tag{2.11}$$

The expression on the r.h.s. of (2.9) takes an elegant form when expressed in terms of the familiar Hermite functions; this is done in paper I where the connection to the Schrödinger approach is also provided. At this juncture it is pertinent to note that (2.9) has a limit as $t \to \infty$ (on the understanding $\omega \to \omega - \mathrm{i}\epsilon$) given by

$$\lim \pi(x, t|x_0) = \pi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega x^2}{\hbar}\right) \tag{2.12}$$

which represents the complex measure density of the coordinate corresponding to the ground state. In this case the measure density is positive definite coinciding with the modulus measure density. It also follows that the ground state is stationary in the strict probability sense which can be verified by multiplying both sides of (2.10) by $\pi(x_0)$ and integrating over x_0.

We next consider the CMD corresponding to the choice (2.8); this corresponds to the displaced oscillator which occupies a pivotal position in coherent state theory. A detailed analysis of the F-P equation and its connection to the coherent theory is provided in [24] (referred to as III in what follows); the conditional CMD is given by

$$f_2(x, t|x_0, t_0) = \left(\frac{m\omega}{\pi\hbar[1 - \mathrm{e}^{-2\mathrm{i}\omega t}]}\right)^{\frac{1}{2}} \exp\left\{-\frac{m\omega}{\hbar}[(x - \alpha)\mathrm{e}^{\mathrm{i}\omega t} - (x_0 - \alpha)]^2/(\mathrm{e}^{2\mathrm{i}\omega t} - 1)\right\} \tag{2.13}$$

where we have used t in the place of $t - t_0$ for notational convenience. The stationary solution which is also the limit as $t \to \infty$ of f_2 (under $\omega \to \omega - \mathrm{i}\epsilon$) takes the form

$$\lim f_2 \equiv \pi(x, \alpha) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega}{\hbar}(x - \alpha)^2\right). \tag{2.14}$$

We note that π is still a genuine CMD. Setting $m = 1$ we note that the coherent state wave function [12,29] can be obtained by taking the square root of CMD with the replacement $\alpha \to \left(\frac{2\hbar}{\omega}\right)^{1/2} \alpha$. In what follows we choose $\hbar = \omega = 1$ and introduce $\phi_\alpha(x)$ by

$$\phi_\alpha(x) = N\pi(x, \alpha)\ , \tag{2.15}$$

$$\pi(x, \alpha) = \left(\frac{1}{\pi}\right)^{1/2} \exp -(x - \alpha\sqrt{2})^2\ , \tag{2.16}$$

where N is so chosen to render the total modulus measure unity:

$$N = \exp -\left(2(\mathrm{Im}\alpha)^2\right). \tag{2.17}$$

It is worth noting that the square root $\phi_\alpha^{1/2}(x)$ of $\phi_\alpha(x)$ is an element of the L_2-space mentioned earlier. We render it a Hilbert space through the inner product corresponding to two CMD's; if f and g are any two CMD's, the inner product is defined by

$$(f^{1/2}, g^{1/2}) = \int f^{1/2}(x)\overline{g^{1/2}(x)}\mathrm{d}x \; . \tag{2.18}$$

Thus if two elements corresponding to coherent states are labelled by α and β, then the inner product is defined by

$$(\phi_\beta^{1/2}, \phi_\alpha^{1/2}) = \exp\left\{\overline{\alpha}\beta - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \mathrm{i}\alpha_1\alpha_2 - \mathrm{i}\beta_1\beta_2\right\} \tag{2.19}$$

in agreement with the Glauber–Sudarshan formula except for the phase factors. The occurrence of the phase factors is a unique feature of the CMD's in CMTF following from the basic assumption that the total complex measure $\lambda(\Omega)$ is unity. It should be specially noted that the operators in CMTF arise from Hilbert space vectors and possible transformations and have no other physical meaning. We will use the notation $|\alpha\rangle$ to denote the vector and interpret it as $\phi_\alpha^{1/2}$ through the definition

$$\langle x|\alpha\rangle = \phi_\alpha^{1/2}(x) \; . \tag{2.20}$$

In the more general case if f is any CMD, the element $f^{1/2}$ of the Hilbert space is denoted by $|f^{1/2}\rangle$ with the definition

$$\langle x|f^{1/2}\rangle = f^{1/2}(x) \; . \tag{2.21}$$

Equally well we can have the momentum representation by dealing with the Fourier transform of $f^{1/2}(x)$; as explained in II, Plancharel theorem ensures that this leads to a properly normalized modulus measure density function in the momentum representation. It is also to be noted that in CMTF there is no scope for the introduction of Fock space as in the standard operator theory; however since the Hermite functions ϕ_n belong to this Hilbert space, we can deal with the projections on the same. For instance we can define

$$\langle n|\beta\rangle = (\phi_\beta^{1/2}, \phi_n) \tag{2.22}$$

where the r.h.s can be evaluated using the definition of the inner product. Next we note that the outer product $|\alpha\rangle\langle\alpha|$ can be formed and through it the concept of density matrix for the pure state introduced:

$$\rho = |\alpha\rangle\langle\alpha| \; . \tag{2.23}$$

The diagonal element corresponding to the coordinate x or momentum p is easily identified as the modulus measure density (MMD) function; moreover the r.h.s of (2.23) when integrated over the complex plane α can be shown to lead to the identity element by simply taking the matrix element of ρ corresponding to (β, γ) where β and γ are arbitrary complex parameters. A more general version of the density matrix for mixed state is introduced by

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha \tag{2.24}$$

where P is an arbitrary complex-valued function of α. In the general setting of CMTF there need be no restriction on P except that it be measurable with total measure equal to unity. Thus the tinge of classicism that is generally ascribed to the density matrix (see in this context [21]) can be removed by keeping $P(.)$ to be genuinely complex-valued. However to be in conformity with the standard operator theory we can restrict it to be real-valued to render it Hermitian. Of course it is a moot question in CMTF whether any purpose is served in keeping $P(.)$ more general, since it is necessary in order to define expectation values, to impose a change of measure leading to positive definiteness of the P-function. Finally we note that while the diagonal elements are easily identified as the MMD's, the non-diagonal elements are products of projections. In CMTF these projections are easily interpreted as weight coefficients of appropriate complex measures and hence can be identified as elements that go to define conditional measures/measure densities.

It is worth noting that superpositions of independent streams can be handled in this formalism. For instance the P-function of the stream corresponding to two independent streams with P-functions $P_1(.)$ and $P_2(.)$ is just the convolution of $P_1(.)$ and $P_2(.)$. To illustrate we consider a simple optical phenomena when a light beam corresponds to the superposition of an arbitrary stream with a thermal stream of light. The problem had been dealt with by Loudon and Shepherd [20] using operator theory. The statistical properties of the photon number in the resulting stream can be nicely handled in the present formalism; in order to maintain the continuity of the discussion, the details of the analysis are relegated to appendix A.

Next we note that it is possible to introduce the concept of a conditional density matrix. This is best done by modifying the P-function. Thus if P is generalized to be a function of α and α_0 we can define the conditional density matrix ρ_c by

$$\rho_c(\alpha_0) = \int P(\alpha, \alpha_0)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha \, . \tag{2.25}$$

A simple example of such a conditioning is provided by

$$P(\alpha, \alpha_0) = \frac{1}{\pi} \exp\left(-\frac{|\alpha - \alpha_0|^2}{N}\right) \tag{2.26}$$

where N is a positive-valued parameter. This corresponds to the coherent state representation of (amplitude) mixture of thermal and coherent light beams, the parameter N being interpreted as the expected number of quanta in the thermal stream. The idea that the density matrix can be made conditional is due to Accardi [1]; in our approach the coherent state representation renders the formulation rather direct and simple. For instance if we have a superposition of a thermal stream and a stream with arbitrary characteristics specified by the P-function $P_1(\alpha_0)$, then the P-function corresponding to the superposed stream is obtained by multiplying the r.h.s of (2.26) by $P_1(\alpha_0)$ and integrating over α_0, a process which is very natural in CMTF as contrasted with the standard operator theory where the superposition principle has to be invoked. An example of a conditional structure which has no parallel in the operator formalism is provided in the next section in the form of a conditional Wigner distribution.

3 Holomorphic extension

So far the discussion centred round a complex measurable process $\{X(t)\}$ and the CMD's are with respect to a real valued random variable like coordinate or momentum. In other words we associate a complex measure density for the random variable in question or more precisely a complex measure for Borel subsets of the real line. It is quite natural therefore to explore the possibility of assigning a complex measure density for the complex coordinate. We have already stated that at the level of Fokker–Planck equation (2.5) extension is possible to cover the case when $\{X(t)\}$ is a vector process. In the theory of classical stochastic processes, complex Gaussian processes are studied (see for example [13]) on the understanding that the real and imaginary parts are independent and identically distributed. The primary motivation for the strong condition is just to arrive at the properties of complex Gaussian systems very similar to those of real Gaussian systems. It turns out that when such a strong condition is imposed in CMTF, we obtain an interesting class of processes that are capable of describing a variety of quantum phenomena. For brevity we call such a complex measure theoretic framework a holomorphic version of complex measure theoretic framework (HCMTF).

As in the previous section we will be mainly interested in harmonic oscillators. If $\{Z(t)\}$ is the stochastic process where for each t, $Z(t) = X(t) + \mathrm{i}Y(t)$ we are interested in the conditional measure density function $\pi(z, t|z_0)$ of $Z(t)$ under the condition $Z(0) = z_0$. As before we assume that $\{Z(t)\}$ is a Markov process satisfying the conditions

$$\lim_{\Delta \to 0} E[Z(t+\Delta) - Z(t)|Z(t) = z]/\Delta = a(z) , \tag{3.1}$$

$$\lim_{\Delta \to 0} E[\{Z(t+\Delta) - Z(t)\}\{\overline{Z}(t+\Delta) - \overline{Z}(t)\}|Z(t) = z]/\Delta = 4D . \tag{3.2}$$

As before we are interested in the two cases

$$\text{(i)} \;\; a(z) = -\mathrm{i}\,\omega , \tag{3.3}$$

$$\text{(ii)} \; a(z) = -i\omega\,(z \; - \; \alpha) , \tag{3.4}$$

which model respectively the free and displaced harmonic oscillator. The diffusion constant $2D$ is set equal to $\mathrm{i}\hbar/m$ as in the earlier case. The CMD $\pi(z, t|z_0)$ corresponding to the free harmonic oscillator now satisfies the Fokker-Planck equation

$$\frac{\partial \pi(z,t|z_0)}{\partial t} = \frac{\partial}{\partial z}(\mathrm{i}\omega z \pi(z,t|z_0)) + \frac{\partial}{\partial \overline{z}}(\mathrm{i}\omega \overline{z} \pi(z,t|z_0)) + 4D \frac{\partial^2 \pi(z,t|z_0)}{\partial z \partial \overline{z}} . \tag{3.5}$$

The above equation is best solved by the method of characteristics and we have

$$\pi(z,t|z_0) = \frac{m\omega}{\pi\hbar} \exp\left(-\frac{m\omega}{h} \frac{(z\mathrm{e}^{\mathrm{i}\omega t} - z_0)(\overline{z}\mathrm{e}^{\mathrm{i}\omega t} - \overline{z}_0)}{(\mathrm{e}^{2\mathrm{i}\omega t} - 1)}\right). \tag{3.6}$$

The only stationary state, as before, is the ground state and is the limit as $t \to \infty$ of $\pi(z, t|z_0)$ (under $\omega \to \omega - \mathrm{i}\epsilon$) given by

$$\pi_H(z) = \frac{m\omega}{\pi\hbar} \exp\left(-\frac{m\omega}{\hbar}|z|^2\right). \tag{3.7}$$

If we set $m = 1$ and identify the coordinate and momentum by

$$X = Q(\text{ coordinate}), \;\; mY = P(\text{ momentum}) , \tag{3.8}$$

we can rewrite (3.7) in the form

$$\pi_H(\hat{z}) = \frac{2}{\pi} \exp -|\hat{z}|^2 , \tag{3.9}$$

where

$$\hat{X} = Q\left(\frac{\omega}{2\hbar}\right)^{1/2}, \qquad \hat{Y} = \frac{P}{(2\hbar\omega)^{1/2}}, \tag{3.10}$$

and $\{\hat{Z}(t)\}$ denotes the stochastic process with stationary CMD $\pi_H(\hat{z})$ which now coincides with the Wigner distribution with variances of the coordinate and momentum normalized to $\frac{1}{4}$. From now on we write z in the place of $\hat{z}$ for notational convenience.

Next we note that the displaced oscillator defined by the usual diffusion function and drift by (3.4) can be handled in exactly the same way. The solution in explicit form for the CMD can be obtained from the expression on the r.h.s. of (3.6) by the replacement $z \to z - \alpha$ and $z_0 \to z_0 - \alpha$. The stationary solution corresponds to the displaced ground state and the CMD is given by

$$_{\mathrm{sty}}\pi_H(z) \equiv \pi_H(z,\alpha) = \frac{1}{\pi} \exp\left(-(z - \alpha\sqrt{2})^2\right) \tag{3.11}$$

where we have preferred to normalize the variance to the original level $\frac{1}{2}$. If on the other hand we use the normalized variables (3.10) as the basis for discussion, then the CMD takes the Wigner form

$$\pi_H(z,\alpha) = \frac{2}{\pi}\exp\left(-2|z-\alpha|^2\right). \tag{3.12}$$

If at this stage we make the transformation

$$z = \mu\zeta + \nu\overline{\zeta}, \mu = \cosh s, \nu = \mathrm{e}^{\mathrm{i}\theta}\sinh s \tag{3.13}$$

where s is an arbitrary real parameter, then using z in the place of ζ for convenience, we have

$$\begin{aligned}\pi_H(z,\beta) &= \frac{2}{\pi}\exp[-2|\mu(z-\beta)+\nu(\overline{z}-\overline{\beta})|^2] \\ &= \frac{2}{\pi}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mathrm{Rl}\beta)^2}{\sigma_X^2}+\frac{(y-\mathrm{Im}\beta)^2}{\sigma_Y^2}\right.\right. \\ &\qquad\left.\left. -2\rho\frac{(x-\mathrm{Rl}\beta)^2(y-\mathrm{Im}\beta)^2}{\sigma_X\sigma_Y}\right\}\right]\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\beta &= \mu\alpha - \nu\overline{\alpha}\,, \\ \sigma_X^2 &= \frac{1}{4}(\mu-\nu)(\mu-\overline{\nu}) = \frac{1}{4}\left(\mathrm{e}^{2s}\sin^2\frac{\theta}{2}+\mathrm{e}^{-2s}\cos^2\frac{\theta}{2}\right)\,, \\ \sigma_Y^2 &= \frac{1}{4}(\mu+\nu)(\mu+\overline{\nu}) = \frac{1}{4}\left(\mathrm{e}^{2s}\cos^2\frac{\theta}{2}+\mathrm{e}^{-2s}\sin^2\frac{\theta}{2}\right)\,, \\ \rho &= -\frac{\sinh 2s\sin\theta}{4\sigma_X\sigma_Y}\,.\end{aligned} \tag{3.15}$$

The transformation (3.13) is generally used on the quadrature variates of a coherent stream of light, the transformation converting the coherent stream into a squeezed coherent stream. The expression as given by (3.14) had been derived by Dodunov et al. [5] who identify the resulting state as a correlated state. The results relating to variance and other properties were obtained by Caves [3] and Yuen [35]; we have preferred to express the final form (3.14) in the notation of Loudon and Knight [19]. Now from the CMTF point of view it is worth noting that the holomorphic version of the CMD of the displaced ground state (oscillator) goes over into the CMD of the squeezed coherent state under the transformation (3.13). The variances of the real and imaginary parts of the complex process are no longer equal, although their product remains invariant under the transformation. This result acquires profound significance from the quantum phenomenal point of view since squeezed streams can be generated physically by parametric amplifiers [7]. For instance the transformation (3.13) induces a measure transformation on the CMD; thus the interaction implied by the parametric amplifier is just a manifestation of the complex probability measure transformation. It is interesting to compare this result with the one in an earlier contribution [26] wherein the Dirac equation of the electron in a uniform magnetic field is derived in the quaternionic measure theoretic framework; in such a situation it turns out that a transformation of the quaternions induces a transformation on the quaternionic measure density which in turn eliminates the magnetic field.

Next we consider the collection of CMD's of complex- valued random variables $\{Z\}$. These random variables need not be restricted to the class introduced in the beginning of the section, since they can be made more general by a transformation of the type (3.13) or other types of transformations. We call this process of enlarging the collection of CMD's holomorphic extension and the resulting measure theoretic framework holomorphic CMTF (HCMTF). The square roots of the CMD's are square integrable and hence they form an L_2-space. The inner product can be introduced in a manner similar to (2.19); we finally render it a Hilbert space by completion. If f and g are two CMD's then the inner product between $f^{1/2}$ and $g^{1/2}$ is defined by

$$(f^{1/2}, g^{1/2}) = \int f^{1/2}(z)\overline{g^{1/2}(z)}\mathrm{d}^2 z \ , \tag{3.16}$$

where $\mathrm{d}^2 z$ stands for $\mathrm{d}x\mathrm{d}y$ and the integration is over the entire (x, y) plane.We shall, as before, use the notation $|f^{1/2}\rangle$ to denote the element of the Hilbert space and define for any $z \in \mathbf{C}$

$$\langle z|f^{1/2}\rangle = f^{1/2}(z) \ . \tag{3.17}$$

Next we consider the collection of CMD's corresponding to the displaced ground state as α varies; the square root functions belong to the Hilbert space. We replace α by $\alpha\sqrt{2}$ in (3.11) and define the element of the Hilbert space corresponding to the displaced ground state CMD to be $|\alpha\rangle$ so that we have

$$\langle z|\alpha\rangle = \pi_H^{1/2}(z, \alpha\sqrt{2}) = (\pi_H(z, \alpha\sqrt{2}))^{1/2} \ . \tag{3.18}$$

It follows that the scalar product between any two elements $|\alpha\rangle, |\beta\rangle$ is given by

$$\langle\beta|\alpha\rangle = (\pi_H^{1/2}(z, \alpha\sqrt{2}), \pi_H^{1/2}(z, \beta\sqrt{2})) = \exp\left(-\frac{1}{2}|\alpha - \beta|^2\right) . \tag{3.19}$$

At this juncture there is a small problem in notation; we have to distinguish between (3.18) and (3.19). While the latter is a scalar product, the former is the square root of the CMD evaluated at z. A similar kind of problem exists even in the standard approach through the Fock space (see for example [16, p. 105]) except for the fact that it is more pronounced in our case. We overcome this by using Roman character to distinguish the square root of the CMD from the scalar product for which we use Greek symbols. In passing we note that the value of the scalar product (3.19) is the same irrespective of the normalization of the variance ($\frac{1}{2}$ or $\frac{1}{4}$).

We next define the density matrix in the holomorphic representation; for the pure state it is simply

$$\rho = |\alpha\rangle\langle\alpha| \ . \tag{3.20}$$

The diagonal element in the complex coordinate representation just gives the Wigner distribution

$$\langle z|\rho|z\rangle = |\langle z|\alpha\rangle|^2 = \frac{2}{\pi}\exp -2|z - \alpha|^2 \tag{3.21}$$

where we have normalized the variance to $\frac{1}{4}$. This result has no parallel in the standard coherent state theory. The density matrix for the general case (mixed state) can be defined as in CMTF by

$$\rho = \int P(\alpha)|\alpha\rangle\langle\alpha|\mathrm{d}^2\alpha \tag{3.22}$$

where $P(.)$ is an arbitrary CMD in α with the only restriction that P be real valued in order that ρ be Hermitian; however it can be taken to be more general without such a restriction as in CMTF. The interesting aspect is that the introduction of P is most natural in HCMTF unlike the situation in CMTF where P enters merely as a weight coefficient. The most remarkable property is that the diagonal element yields the Wigner distribution

$$\langle z|\rho|z\rangle = \frac{2}{\pi}\int P(\alpha)\exp\left(-2|z - \alpha|^2\right)\mathrm{d}^2\alpha \ . \tag{3.23}$$

An important property having no parallel in the standard operator theory is that the r.h.s. is a convolution of P with the Wigner distribution corresponding to the coherent state. It is to be specially noted that many of the computations generally performed to bring out the features of quantum optical phenomena can now be done with extraordinary ease. Normally the P-function does not seem to play a significant role (see for example [35]) in the standard operator theory; this is no longer so in HCMTF. It follows from (3.23) that the complex Fourier transform of the Wigner distribution defined by

$$\tilde{W}(\eta) = \int W(z) \exp(z\overline{\eta} - \overline{z}\eta) \mathrm{d}^2 z \tag{3.24}$$

is now given by

$$\tilde{W}(\eta) = \tilde{P}(\eta) \exp\left(-\frac{|\eta|^2}{2}\right) \tag{3.25}$$

a result well-known in operator theory as the relation connecting the Weyl characteristic function to the Fourier transform of P-function. However in HCMTF, $\tilde{W}(.)$ has no other interpretation than it being the complex Fourier transform of the holomorphic density function.

We consider two simple situations in quantum optical phenomena. The first is the process of thermalizing a squeezed coherent stream of light (see for example [7]). By an extension of the argument used above, it follows that the Wigner distribution is just a convolution of the P-function of the thermal stream and Wigner distribution of the squeezed vacuum

$$W(z) = \frac{2}{\pi^2 N} \int \left\{\exp\left(-\frac{|\alpha|^2}{N}\right)\right\} \left\{\exp\left(-2|\xi - \beta|^2\right)\right\} \mathrm{d}^2\alpha \tag{3.26}$$

where

$$\xi = \mu z + \nu\overline{z}\ , \qquad \beta = \mu\alpha + \nu\overline{\alpha}\ . \tag{3.27}$$

The integral is evaluated by a change of variable to β and we obtain

$$W(z) = \frac{2}{\pi\mathcal{A}} \exp -\frac{2}{\mathcal{A}^2}\{[1 + 2N(\mu^2 + |\nu|^2)]|\xi|^2 - 4\mu N(\overline{\nu}\xi^z + \nu\overline{\xi}^2)\} \tag{3.28}$$

where

$$\mathcal{A} = [(2N+1)^2 + 8N|\nu|^2]^{1/2}\ . \tag{3.29}$$

The thermalization of vacuum is manifested in the r.h.s. of (3.26). If on the other hand we wish to thermalize a squeezed coherent stream, all that we need do is to replace α by $\alpha - \alpha_0$ where α_0 is the parameter characterizing the coherent stream. The integrand also shows in a transparent manner that thermalization is just a process of superposition of a thermal stream with squeezed vacuum. The P-function can be identified by an inspection of the integrand on the r.h.s. of (3.26) or by using the relation (3.24). After some straight forward evaluation, we obtain

$$P(\alpha) = \frac{1}{\pi[Q(N)]^{1/2}} \exp\left[\left\{-(N + |\nu|^2)|\alpha - \alpha_0|^2 - \frac{\mu}{2}(\nu[\overline{\alpha} - \overline{\alpha_0}]^2 + \overline{\nu}[\alpha - \alpha_0]^2)\right\}/Q(N)\right] \tag{3.30}$$

where

$$Q(N) = N^2 + (2N-1)|\nu|^2\ . \tag{3.31}$$

The P-function as displayed above also manifestly shows its singular nature when $N = 0$; the characteristics corresponding to the squeezed coherent stream can be obtained from the above representation by explicit use of it and then passing to the limit as $N \to 0$. Thus we have an essentially physical way of arriving at a representation of the squeezed coherent P-function which exists in a generalized sense.

The next interesting optical phenomena is the process of squeezing a thermal stream [7]. This is comparatively straight forward; all that we need to do is to start with the Wigner distribution of the thermal stream

$$W(z) = \frac{2}{\pi^2 N} \int \left\{ \exp\left(-\frac{|\alpha|^2}{N} \right) \right\} \left\{ \exp\left(-2|z-\alpha|^2 \right) \right\} \mathrm{d}^2\alpha \ . \tag{3.32}$$

We next note that squeezing in HCMTF is simply making the transformation $z \to \mu z + \nu \bar{z}$; thus we finally have

$$W(z) = \frac{2}{\pi(2N+1)} \exp[-2|\mu z + \nu \bar{z}|^2/(2N+1)] \ . \tag{3.33}$$

All the characteristics of the squeezed thermal light can be on the basis of the above formula. The P-function in this case is given by

$$P(\alpha) = \frac{1}{\pi[Q(N)]^{1/2}} \exp[-\{N+(2N+1)|\nu|^2\}|\alpha|^2 + (2N+1)\mu(\bar{\nu}\alpha^2 + \nu\bar{\alpha}^2)/2\}/Q(N)] \tag{3.34}$$

where

$$Q(N) = N^2 - (2N+1)|\nu|^2 \ . \tag{3.35}$$

We note that the P-function corresponding to the situation when there is a coherent component can be obtained by the replacement $\alpha \to \alpha - \alpha_0$.

So far the discussion centred round the Wigner distribution function; however the photon number is also an interesting characteristic of the stream of light. To arrive at the number distribution we have to resort to indirect methods since CMTF is not operator based. However the holomorphic measure densities can be expanded in terms of complex Hermite functions which besides being a complete orthonormal set belong to the same Hilbert space as the measure densities. In paper III we have provided explicit representation for the displaced ground state oscillator CMD's; however there is a small problem since the representation involves vectors (n_1, n_2). We can use Hida's theorem on the direct sum decomposition of the space (see [13, p. 244]) to arrive at the scalar representation. Thus we can deal with the diagonal elements of the density matrix in this representation; the final result is the same as in CMTF namely that the photon number is conditionally Poisson distributed with parameter $|\alpha|^2$. Using this result it is indeed possible to express the generating function of the photon number distribution in terms of the Wigner distribution (see Eqn. A.11 in appendix A); the generating function $G(u)$ by

$$G(u) = \sum u^n p_n \tag{3.36}$$

for the two cases discussed in the above paragraph are:

(i) Thermalization of squeezed coherent stream:

$$G(u) = \frac{1}{[\mathcal{A}(u)]^{1/2}} \exp\left[\left\{ -|\alpha_0|^2(1-u)[1+(\eta+|\nu|^2)(1-u)] - \frac{\mu}{2}(\bar{\nu}\alpha_0^2 + \nu\overline{\alpha_0}^2)(1-u)^2 \right\} / \mathcal{A}(u) \right] \tag{3.37}$$

where

$$\mathcal{A}(u) = 1 + 2(1-u)(\eta+|\nu|^2) + (1-u)^2(\eta^2 + [2\eta-1]|\nu|^2) \ , \tag{3.38}$$

(ii) Squeezed thermal stream:

$$G(u) = [1 + (u-1)\{[N^2 - (2N+1)|\nu|^2](u-1) - 2[N + (2N+1)|\nu|^2\}]^{-1/2} . \tag{3.39}$$

The first two moments of the photon distribution have been obtained by Fearn and Collet [7] and are in agreement with those obtained from (3.38) and (3.39).

Finally we note that it is possible to introduce a conditional structure on the Wigner distribution. Just as in CMTF, we can replace $P(\alpha)$ by $P(\alpha, \alpha_0)$. Such a conditional Wigner distribution will naturally lead to the notion of conditional density matrix in a natural way. There is yet another way by which we can arrive at the conditional Wigner distribution. We note that (3.6) itself a distribution conditioned by $Z(0) = z_0$. This is rather a trivial example of a conditional Wigner distribution which is genuinely complex-valued. A more illuminating example can be obtained by conditioning the initial state; thus multiplying both sides of (3.6) by $\pi_H(z_0, \alpha)$ as given by (3.11) and integrating over z_0, we obtain

$$\pi_{\mathrm{gdd}}(z, \alpha, t) = \frac{1}{\pi} \exp\left(-\{(z - \alpha \mathrm{e}^{-\mathrm{i}\omega t})(\overline{z} - \overline{\alpha}\mathrm{e}^{-\mathrm{i}\omega t})\}\right) . \tag{3.40}$$

The density matrix that corresponds to this Wigner distribution is easily computed:

$$\rho(x, x', \alpha, t) = \frac{1}{\sqrt{\pi}} \exp\left[-\left\{\left(\frac{x + x'}{2} - \alpha_1 \mathrm{e}^{-\mathrm{i}\omega t}\right)^2 - \mathrm{i}\alpha_2 \mathrm{e}^{-\mathrm{i}\omega t}(x - x') + \left(\frac{x - x'}{2}\right)^2\right\}\right] \tag{3.41}$$

where $\alpha = \alpha_1 + \mathrm{i}\alpha_2$. The function ρ as given by (3.41) is the conditional density matrix and has no parallel in the standard operator theory; needless to emphasize that neither ρ nor π_{gdd} is positive-definite.

4 Test oscillator in a bath

Now we are comfortably placed to discuss the various aspects of a test (harmonic) oscillator in interaction with a collection of oscillators consisting a bath. Normally the study of interacting oscillator is a complex one; however in our case we make the (drastic) bath approximation so that the test oscillator feels the effects of the bath substantially with the bath itself being unaffected by the presence/interaction of the test oscillator. Such an oscillator had been the subject of investigation from various angles by numerous authors including Feynman and Vernon [8], Ford et al. [9] and Caldeira and Legget [2]. The model had been elegantly handled by the path integral method by Feynman and Vernon [8] treating the distinguished (test) oscillator as a forced harmonic oscillator, the forcing term arising from its interaction with the bath. Dowker and Halliwell [6] had discussed this problem with special reference to the general aspects of quantum mechanical history and decoherence. We here show how the model can be handled in CMTF and many interesting properties can be brought out in a simple and direct manner.

We note that the motion of the distinguished (test) oscillator in interaction with the bath is best visualized as a forced harmonic oscillator with the bath characteristics included in the forcing term. In CMTF, we can introduce the bath characteristics ($\mathbb{B}$) as some kind of a conditioning; denoting the conditional CMD by $\pi(x, t|x_0, t_0; \mathbb{B})$ we note that π satisfies the Fokker–Planck equation (2.5) where $b(x)$ is the usual complex diffusion coefficient $\mathrm{i}\hbar/m$ and $a(x)$ which is now a function of t also is specified by

$$a(x) = -\mathrm{i}\omega x + \beta(t) , \tag{4.1}$$

$$\beta(t) = \int_0^t \mathrm{e}^{-\mathrm{i}\omega(t-s)} f(s) \mathrm{d}s , \tag{4.2}$$

$$f(t) = -\sum_k C_k R_k(t) . \tag{4.3}$$

The additional term $\beta(t)$ in the drift function $a(x)$ brings out in a transparent manner the linear coupling (of the distinguished oscillator) with the coordinate $R_k(t)$ of the typical oscillator of the bath. The Fokker–Planck equation (2.5) for the choice of the choice of the drift function $a(x)$ for a very general $\beta(.)$ has been analyzed in detail in papers I and II in connection with the classic problem of interaction of light with matter; for our present discussion we just take over the solution and consider it in detail when $f(.)$ is specified by (4.3). Thus the conditional CMD $\pi(x,t|x_0,t_0;\mathbb{B})$ is given by

$$\pi(x,t|x_0,t_0;\mathbb{B}) = \left(\frac{m\omega \mathrm{e}^{2\mathrm{i}\omega t}}{\pi\hbar(\mathrm{e}^{2\mathrm{i}\omega t}-1)}\right)^{1/2} \exp\left\{-\frac{m\omega \mathrm{e}^{2\mathrm{i}\omega t}}{\mathrm{e}^{2\mathrm{i}\omega t}-1}\left[x - x_0\mathrm{e}^{-\mathrm{i}\omega t} - F(t,t_0)\right]^2\right\} \tag{4.4}$$

where

$$F(t,t_0) = \frac{1}{\omega}\int_{t_0}^{t} f(u)\sin\omega(t-u)\mathrm{d}u\ . \tag{4.5}$$

In the ultimate analysis we consider a continuous assembly of oscillators constituting the bath; however to obtain results in a rather simple way, we note that if we deal with one oscillator, then $f(u)$ in (4.5) can be replaced by $-CR(u)$ whose CMD has a Gaussian structure. The process of summing over the different oscillators constituting the bath is analyzed in Appendix B where details of analysis leading to the determination of the statistical characteristics of $F(t,t_0)$ are given; the final result is given by

$$\begin{aligned}\pi_{\mathrm{final}}(x,t|x_0,t_0) &\equiv \mathrm{E}_{\mathbb{B}}[\pi(x,t|x_0,t_0;\mathbb{B})]\\ &= \left\{2\pi\left[\mathcal{A} + \frac{\hbar}{2m\omega}(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})\right]\right\}^{-1/2} \times\\ &\quad\times \exp - \left(x - x_0\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}\right)^2 \Big/ \left[2\mathcal{A} + \frac{\hbar}{m\omega}\left(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)}\right)\right]\end{aligned} \tag{4.6}$$

where $\mathrm{E}_{\mathbb{B}}$ denotes the expectation over the bath variables and $\mathcal{A}$ is given by

$$\mathcal{A} = \sum_n \frac{C_n^2\hbar}{M\Omega_n\omega^2}\left[I_1(n) + \left(\coth\frac{\Omega_n\hbar}{2kT} - 1\right)I_2(n)\right]\ . \tag{4.7}$$

The above expression is for the most general case corresponding to independent evolution of the oscillators of the bath, each starting from a thermal equilibrium state with temperature T. If however we use the same approximation as the one used by Caldiera-Legget or Dowker and Halliwell [6] then r.h.s. simplifies considerably since the oscillators of the bath are assumed to be in thermal equilibrium for all time; in such a case

$$I_1(n) = I_2(n) = \frac{2}{\omega^2}\sin^2\frac{\omega}{2}(t-t_0)\ . \tag{4.8}$$

Following Caldeira and Legget if we choose a continuum of oscillators with density $\rho_D(\Omega)$, we simply make the replacement

$$\sum_n \to \int \mathrm{d}\Omega\rho_D(\Omega), C_n \to C(\Omega)$$

so that we finally have

$$\mathcal{A} = \frac{2\hbar}{M\omega^4}\sin^2\omega(t-t_0)\int_0^\infty C^2(\Omega)\rho_D(\Omega)\coth\frac{\Omega\hbar}{2kT}\mathrm{d}\Omega\ . \tag{4.9}$$

The complex representation of $\mathcal{A}$ even in its most general form (4.7) provides an easy interpretation of the main characteristics of the test oscillator; besides renormalization of the frequency the dissipation due to bath is apparent from (4.6). Next we note that (4.6) corresponds to the situation when the test oscillator is initially constrained at a fixed position x_0; if on the other hand we take the initial configuration to correspond to the coherent state α, then defining

$$p(x,t|\alpha,t_0) = \pi(x,t|\alpha,t_0)|_{\text{normalized}} \;, \tag{4.10}$$

$$\pi(x,t|\alpha,t_0) = \int \pi_{\text{final}}(x,t|x_0,t_0)\left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left[-\frac{m\omega}{\hbar}(x_0-\alpha)^2\right] \mathrm{d}x_0 \;, \tag{4.11}$$

we find that $p(x,t|\alpha,t_0)$ can be expressed neatly in terms of the phases ϕ, γ respectively of the complex parameters $\frac{\hbar}{2m} + \mathcal{A}, \alpha$;

$$p(x,t|\alpha,t_0) = \left(\frac{N}{\pi}\right)^{1/2} \exp\left(-\frac{\cos\phi}{2|\frac{\hbar}{2m\omega}+\mathcal{A}|}\left\{x - |\alpha|\frac{\cos[\omega(t-t_0)+\phi+\gamma]}{\cos\phi}\right\}^2\right), \tag{4.12}$$

where

$$N = \cos\phi / \left[2|\frac{\hbar}{2m\omega} + \mathcal{A}|\right] \;. \tag{4.13}$$

Next we provide an explicit expression for the two time CMD of the test oscillator which is conditioned to be in a coherent state labelled by α. We note that the test oscillator has Markov character conditional on bath (variables); introducing $\pi(x_2,t_2;x_1,t_1|\alpha,t_0;\mathbb{B})$ as the conditional CMD that $X(t)=x_1, X(t)=x_2$ we find

$$\pi(x_2,t_2;x_1,t_1|\alpha,t_0;\mathbb{B}) = \pi(x_1,t_1|\alpha,t_0;\mathbb{B})\pi(x_2,t_2|x_1,t_1;\mathbb{B}) \;. \tag{4.14}$$

Making repeated use of (4.4) and (4.11) and using the general arguments presented in Appendix B particularly the Gaussian nature of the bath coordinates we note

$$\begin{aligned} \pi(x_2,t_2;x_1,t_1|\alpha,t_0) &= \mathrm{E}_{\mathbb{B}}[\pi(x_2,t_2;x_1,t_1|\alpha,t_0)] \\ &= \int \pi(x_2,t_2;x_1,t_1|\alpha,t_0;\mathbb{B}) \frac{1}{\sqrt{2\pi\mathcal{A}\mathcal{B}-\mathcal{C}^2}} \times \\ &\quad \times \exp - \frac{\mathcal{A}\mathcal{B}}{2(\mathcal{A}\mathcal{B}-\mathcal{C}^2)}\left\{\frac{z^2}{\mathcal{A}} + \frac{w^2}{\mathcal{B}} - 2\frac{\mathcal{C}zw}{\mathcal{A}\mathcal{B}}\right\} \mathrm{d}z\mathrm{d}w \end{aligned} \tag{4.15}$$

where the random variables (Z, W) corresponding to (z, w) are given by

$$Z = F(t_1,t_0), W = F(t_2,t_1) \tag{4.16}$$

where

$$\mathcal{A} = \mathcal{A}(t,t_0), \mathcal{B} = \mathcal{A}(t_2,t_1) \;. \tag{4.17}$$

The explicit expressions of $\mathcal{A}$ follow from (4.7) or (4.9) depending on the level of approximation employed.The function $\mathcal{C}$ is little bit complex in its structure; nevertheless an expression for the same can be obtained in an exactly same way as $\mathcal{A}$. However if we assume that the bath variables are in thermal equilibrium for all times, then

$$\mathcal{C} = \sum_n \frac{2\hbar C_n^2}{\omega^4 M\Omega}\sin^2\omega(t_2-t_1)\sin^2\omega(t_1-t_0)\coth\frac{\Omega\hbar}{2kT} \;. \tag{4.18}$$

On performing the integration over (z, w) we finally obtain

$$\pi(x_2, t_2; x_1, t_1|\alpha, t_0) = \frac{(N_1N_2)^{1/2}}{\pi[(2N_1\mathcal{A}+1)(2N_2\mathcal{B}+1) - 4N_1N_2\mathcal{C}^2]^{1/2}} \times$$
$$\times \exp - \frac{(\mathcal{A}+\frac{1}{2N_1})(\mathcal{B}+\frac{1}{2N_2})}{2[(\mathcal{A}+\frac{1}{2N_1})(\mathcal{B}+\frac{1}{2N_2}) - C^2]} \left\{ \frac{Q^2}{\mathcal{B}+2N_2} + P^2 \left[\frac{1}{(\mathcal{A}+\frac{1}{2N_1})} + \frac{\mathrm{e}^{-2\mathrm{i}\omega(t_2-t_1)}}{(\mathcal{B}+\frac{1}{2N_2})} \right.\right.$$
$$\left.\left. + \frac{2C\mathrm{e}^{-\mathrm{i}\omega(t_2-t_1)}}{(\mathcal{A}+\frac{1}{2N_1})(\mathcal{B}+\frac{1}{2N_2})} \right] - 2PQ \left[\frac{C}{(\mathcal{A}+\frac{1}{2N_1})(\mathcal{B}+\frac{1}{2N_2})} + \frac{\mathrm{e}^{-\mathrm{i}\omega(t_2-t_1)}}{\mathcal{B}+\frac{1}{2N_2}} \right] \right\} \quad (4.19)$$

where

$$P = x_1 - \alpha\mathrm{e}^{-\mathrm{i}\omega(t_1-t_0)}, Q = x_2 - \alpha\mathrm{e}^{-\mathrm{i}\omega(t_2-t_0)} ,$$
$$N_1 = \frac{m\omega}{\hbar}, N_2 = \frac{m\omega}{\hbar(1-\mathrm{e}^{-2\mathrm{i}\omega(t_2-t_1)})} . \quad (4.20)$$

All the relevant properties of the correlation function can be readily inferred from the above expression.

We finally attempt to find an explicit expression for the Wigner distribution using the idea of holomorphic extension of CMTF. We start with the forced harmonic oscillator in complex coordinate representation; all that we need to do is to modify the expression for drift as given by (3.3) or (3.4). We simply take the complexified version of (4.1)

$$a(z) = -\mathrm{i}\omega(z) + \beta(t) \quad (4.21)$$

where $\beta(t)$ is still given (4.2) and (4.3); $R_k(t)$ now takes the complexified form:

$$R_k(t) = P_k(t) + \mathrm{i}Q_k(t) . \quad (4.22)$$

The diffusion function as defined by (3.2) is set equal to $\mathrm{i}\hbar/m$ as in the earlier case. The resulting Fokker Planck equation is solved in exactly the same manner as (3.5) and we finally obtain

$$\pi(z, t|z_0, t_0; \mathbb{B}) = \frac{m\omega}{\pi\hbar(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})} \times \quad (4.23)$$
$$\times \exp\left[-\frac{m\omega}{\hbar(\mathrm{e}^{2\mathrm{i}\omega t} - \mathrm{e}^{2\mathrm{i}\omega t_0})} (z\mathrm{e}^{\mathrm{i}\omega t} - z_0\mathrm{e}^{\mathrm{i}\omega t_0} - \int_{t_0}^{t} \beta(s)\mathrm{e}^{\mathrm{i}\omega s}\mathrm{d}s)(\overline{z}\mathrm{e}^{\mathrm{i}\omega t} - \overline{z_0}\mathrm{e}^{\mathrm{i}\omega t_0} - \int_{t_0}^{t} \overline{\beta(s)}\mathrm{e}^{\mathrm{i}\omega s}\mathrm{d}s) \right] .$$

The average over the bath variables is easily done by noting that the solution as given by (3.6) in the previous section describes the evolution of the typical bath variable; all that we need to do is to modify the initial condition so that each variable is in thermodynamic equilibrium at temperature T; thus the solution as given by (3.6) when averaged over x_0 takes the form

$$\pi(z, t| \text{ thermal equilibrium})$$
$$= \frac{M\Omega}{\pi\hbar} \frac{1}{[1-\mathrm{e}^{-2\mathrm{i}\Omega(t-t_0)}](1-\coth\frac{\Omega\hbar}{2kT})} \exp -|z|^2 \frac{\Omega M}{\pi[1-\mathrm{e}^{-2\mathrm{i}\Omega(t-t_0)}](1-\coth\frac{\Omega\hbar}{2kT})} . \quad (4.24)$$

We next note that the averaging over the bath variables follow exactly in the same manner as in the case of real random variable R_k; thus we have an expression analogous to (4.6):

$$\pi_{\text{final}}(z, t|z_0, t_0) = \mathrm{E}_{\mathbb{B}}[\pi(z, t|z_0, t_0; \mathbb{B})]$$
$$= \frac{1}{2\pi[\mathcal{A} + \frac{\hbar}{2m\omega}(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})]} \exp - \frac{(z - z_0\mathrm{e}^{-\mathrm{i}\omega(t-t_0)})(\overline{z} - \overline{z_0}\mathrm{e}^{-\mathrm{i}\omega(t-t_0)})}{2[\mathcal{A} + \frac{\hbar}{2m\omega}(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})]} \quad (4.25)$$

where $\mathcal{A}$ is the state given by (4.7). Thus we have an interesting result; if the initial condition for the test oscillator is specified by the Wigner distribution function $w(\overline{z_0}, z_0)$ (this fits in very well with HCMTF since this is just the initial condition for the holomorphic density for the coordinate) then the final holomorphic density for the test oscillator which is interpreted to be the Wigner distribution function is now given by

$$\pi^H_{\text{testoscillator}}(z,t|t_0) = \int \frac{\mathrm{d}^2 z_0 w(\overline{z_0}, z_0)}{2\pi[\mathcal{A} + \frac{\hbar}{2m\omega}(1 - \mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})]} \times \tag{4.26}$$

$$\times \exp\left\{-[|z|^2 + |z_0|^2 \mathrm{e}^{-2\mathrm{i}\omega(t-t_0)} - (z_0\overline{z} + \overline{z_0}z)\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}]/2[\mathcal{A} + \frac{\hbar}{2m\omega}(1 - \mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})]\right\} .$$

This is the HCMTF analogue of equation (5.44) of Dowker-Halliwell [6]. If we express $w(\overline{z_0}, z_0)$ in terms of its complex Fourier Transform $Q(\overline{\eta}, \eta)$ the above result can be put in a more compact form:

$$\pi^H_{\text{testoscillator}}(z,t|t_0) \tag{4.27}$$
$$= \frac{1}{\pi}\int Q(\overline{\eta},\eta)\exp\left\{\mathrm{e}^{\mathrm{i}\omega(t-t_0)}(z\overline{\eta} - \overline{z}\eta) - 2[\mathcal{A} + \frac{\hbar}{2m\omega}(1 - \mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})]|\eta|^2 \mathrm{e}^{2\mathrm{i}\omega(t-t_0)}\right\}\mathrm{d}^2\eta .$$

The above result can be the basis for further discussion of correlation structure of the test oscillator. In an analogous way the two time correlation of the coordinate of the test oscillator can be arrived at. However it should be noted that modulus measure of the two time distribution does not exist in the ordinary sense since this is one of the cases where the variational measure is to be interpreted in a generalized sense [15]. In this context it is worth noting that no such correlation can exist from an observational point of view and at best it can only be defined in a sequential way which takes into account the interference effects.

Next we briefly discuss the method of arriving at probabilities of paths in a situation where the oscillator is constrained to pass through a couple of gaussian slits. We start with (4.11); defining

$$p(x,t|\alpha,t_0) = \pi(x,t|\alpha,t_0)|_{\text{normalized}} \tag{4.28}$$

$$\pi(x,t|\alpha,t_0) = \int \pi_{\text{final}}(x,t|x_0,t_0)\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left[-\frac{m\omega}{\hbar}(x_0-\alpha)^2\right]\mathrm{d}x_0 , \tag{4.29}$$

we find that $p(x_1, t_1|\alpha, t_0; \mathbb{B})$ is given by

$$\begin{aligned} p(x_1,t_1|\alpha,t_0,\mathbb{B}) &= \pi(x_1,t_1|\alpha,t_0,\mathbb{B})|_{\text{mod}} \\ &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left\{-\frac{m\omega}{\hbar}[x_1 - \alpha_1\cos\omega(t_1-t_0) - \alpha_2\sin\omega(t_1-t_0) - F(t_1,t_0)]^2\right\} . \end{aligned} \tag{4.30}$$

Next we pass the oscillator through a slit centered at $\overline{x}$ at time t_1 to obtain

$$p(\overline{x},t_1|\alpha,t_0,\mathbb{B}) = \left(\pi\{\sigma^2 + \frac{\hbar}{m\omega}\}\right)^{-1/2}\exp\left\{-\frac{[\overline{x} - F^\alpha(t_1,t_0)]^2}{\sigma^2 + \frac{\hbar}{m\omega}}\right\} . \tag{4.31}$$

We next note that under the assumption that each of the bath oscillators is under thermal equilibrium,a major simplification arises namely the measure density p continues to be positive real valued even after the conditioning of the bath variables are removed by virtue of $\mathcal{A}$ remaining positive real valued(see (4.9)). Thus removing the conditioning,we obtain

$$p(\overline{x},\sigma,t_1|\alpha,t_0) = \left(\pi\{\sigma^2 + \frac{\hbar}{m\omega}\}\right)^{-1/2}\int(2\pi\mathcal{A})^{-1/2}\exp\left(-\frac{z^2}{\mathcal{A}}\right)\times$$
$$\times\exp\left\{-\frac{[\overline{x} - \alpha_1\cos\omega(t_1-t_0) - \alpha_2\sin\omega(t_1-t_0) - z]^2}{\sigma^2 + \frac{\hbar}{m\omega}}\right\}\mathrm{d}z$$

$$= \left(\pi[2\mathcal{A} + \sigma^2 + \frac{\hbar}{m\omega}] \right)^{1/2} \exp\left\{ -\frac{[\overline{x} - \alpha^1 \cos\omega(t_1 - t_0) - \alpha_2 \sin\omega(t_1 - t_0)]^2}{2\mathcal{A} + \sigma^2 + \frac{\hbar}{m\omega}} \right\}. \tag{4.32}$$

It is interesting to note that the function $p(., \sigma, t_1)$ is not a density in as much as the probabilities corresponding to two different but close enough values of $\overline{x}$ are not exclusive; nevertheless the integral of $p(\overline{x}, \sigma, t_1)$ with respect to $\overline{x}$ yields unity by virtue of the special form of the function $f_W(.)$. In conventional quantum mechanical approach as contrasted with CMTF, this aspect goes unnoticed in as much as all vectors in the Hilbert space are automatically chosen to have norm equal to unity.[1]We shall presently see the gravity of the situation when we deal with passage through two slits. Anyway the elegant formula (4.32) is one of the significant results following from CMTF and can be used as a test of the theory itself.

It is needless to repeat the remarks made in the earlier section which are quite pertinent in the present context. The probability density function of $X(t_2)$ conditional on the oscillator having passed through the slit centered at $\overline{x}$ at time t_1 can be evaluated; we first arrive at the conditional CMD $\pi(x_2, t_2|\overline{x}, \sigma, t_1; \alpha, t_0, \mathbb{B})$ using elementary arguments:

$$\begin{aligned} &\pi(x_2, t_2|\overline{x}, \sigma, t_1; \alpha, t_0, \mathbb{B}) \\ &= \left(\frac{L}{\pi}\right)^{1/2} \exp\left\{ -L \left(x_2 - F(t_2, t_1) - \frac{e^{-i\omega(t_2-t_1)}}{\frac{m\omega\sigma^2}{\hbar} + 1} \{\overline{x} + \frac{m\omega\sigma^2}{\hbar} + F_\alpha\} \right)^2 \right\} \end{aligned} \tag{4.33}$$

where L is given by

$$L = \frac{m\omega}{\hbar} \; \frac{\frac{m\omega\sigma^2}{\hbar} + 1}{\frac{m\omega\sigma^2}{\hbar} + 1 - e^{-i\omega(t_2-t_1)}} \; . \tag{4.34}$$

Thus the conditional probability is given by

$$p(x_2, t_2|\overline{x}, \sigma, t_1; \alpha, t_0, \mathbb{B}) = \left(\frac{m\omega K}{\pi\hbar}\right)^{1/2} \exp\left\{ -\frac{m\omega K}{\hbar}[x_2 - H_\alpha]^2 \right\} \tag{4.35}$$

where K is defined by

$$K = \frac{\left(\frac{m\omega\sigma^2}{\hbar} + 1\right)\left(\frac{m\omega\sigma^2}{\hbar} + 2\sin^2\omega(t_2 - t_1)\right)}{\left(\frac{m\omega\sigma^2}{\hbar} + 2\sin^2\omega(t_2 - t_1)\right)^2 + 4\sin^2\omega(t_2 - t_1)\cos^2\omega(t_2 - t_1)} \tag{4.36}$$

and H_α is given by

$$H_\alpha = F(t_2, t_1) + \{\overline{x} + \frac{m\omega\sigma^2}{\hbar} F_\alpha\} G \tag{4.37}$$

where again G is defined by

$$G = \left(\frac{m\omega\sigma^2}{\hbar}\right) \cos\omega(t_2 - t_1) \left(\frac{1}{(\frac{m\omega\sigma^2}{\hbar} + 1)(\frac{m\omega\sigma^2}{\hbar} + 2\sin^2\omega(t_2 - t_1))} \right). \tag{4.38}$$

[1] However Dowker and Halliwell [6] did observe the overlapping while introducing projectors for Gaussian slits and had consequently taken into account the overlap while providing constraints for decoherence

It thus follows that the conditional probability of passage through the second slit is given by

$$p(\overline{y}, \rho, t_2|\overline{x}, \sigma, t_1\alpha, t_0, \mathbb{B}) = \left(\pi\{\rho^2 + \frac{\hbar}{m\omega K}\}\right)^{-1/2} \exp\left\{\frac{(\overline{y} - H_\alpha)^2}{\rho^2 + \frac{\hbar}{m\omega K}}\right\}. \tag{4.39}$$

Again we note that despite the fact that the probabilities corresponding to two distinct values of $\overline{y}$ overlap,the function $p(\overline{y}, \rho, t_2|\overline{x}, \sigma, t_1\alpha, t_0, \mathbb{B})$ is normalized in the sense the integral of p over $\overline{y}$ is equal to one. Although the CMD π given by (4.33) can lead to some kind of a complex probability of passage through the slit,it is not a complex measure in a technical sense and hence no variational measure can be extracted out of it and this is the main reason why we first obtained a positive measure density from π and then proceeded to evaluate the probability of transit through the slit. Thus the question of assignment of consistent probabilities does not arise. In fact the probability of passage through the slit as given by (4.39) can be used as a crucial test for CMTF itself. Another noteworthy point is that we have a framework to define in a consistent way a sequence of measurements(corresponding,in this case, to a sequence of passages through the slits) since the interference of each measurement is describable as a conditional measure in the first instance. Again this result can be put to test if the gadenken for the Gaussian slit can be realized physically.

5 Summary and conclusion

In this contribution we have provided a detailed analysis of the various characteristics of the complex density function introduced earlier. In particular we have demonstrated how the analysis through the Hilbert space constructed from the natural L_2-space of the collection of square root of the CMD's enables us to deal with the coherent states and thereby introduce the P-function in a general context. The notion of density matrix in its most general form follows in quite a natural way; since CMTF is probability based, we are able to extend the concept of density matrix under conditional events. The very complex-valued nature of the CMD enables to introduce a holomorphic framework by a natural holomorphic extension of the CMD. Such an extension leads to the Wigner distribution quite generally. The density matrix is introduced in the holomorphic frame work and it turns out that the diagonal element corresponds to the Wigner distribution. This leads to an enormous simplicity in computations and we have demonstrated how quantum optical phenomena can be handled in such a frame work. In particular we have shown how computations leading to the determination of the characteristics of the photon number distribution can be handled; explicit expression for the generating function of the photon number distribution in the case of squeezed thermal light and the stream obtained by thermalizing a squeezed stream. We believe that the new representation for the coherent state and its ramification in the holomorphic frame work may prove to be useful in the general formulation of various problems of quantum phenomena. Finally we have analyzed the Caldeira-Legget model of an oscillator in interaction with a bath and provided explicit expressions for the CMD of the test oscillator. Using the Markov nature of the coordinate in the presence of the bath, obtained an explicit expression for the joint complex measure density of the coordinate at two different time points. We believe that this may pave the way unraveling many of the features relating to observation at one time point and its interference effect at a later time point. We have demonstrated this in a very limited context of passage of the test oscillator through two Gaussian slits, thus providing an interpretation of the probability of paths corresponding to the passage through the slits. The holomorphic extension of the CMD of the test oscillator is an interesting result by itself since it can throw light on the evolution of Wigner distribution starting from a given set of conditions.

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A Appendix

We note that the diagonal element of ρ in the number representation (as defined by (2.22)) represents the modulus measure and hence represents the probability of the corresponding number of photons. Now the P-function corresponding to the superposition of a thermal stream with an arbitrary stream whose P-function is $P_0(.)$ is now given by

$$P(\alpha) = \frac{1}{\pi N} \int P_0(\beta) \mathrm{e}^{-\frac{|\alpha-\beta|^2}{N}} \mathrm{d}^2\beta \ . \tag{A.1}$$

We next note that

$$\langle n|\beta\rangle = \frac{\beta^n}{(n)^{1/2}} \exp\left(-\frac{|\beta|^2}{2} - \mathrm{i}\beta_1\beta_2\right) \tag{A.2}$$

from which it follows that the distribution of the number of photons in a stream is conditionally Poisson with parameter $|\beta|^2$; it also follows that the conditional factorial moment of order n of the number of photons is just $|\beta|^{2n}$. Denoting by N_s, N_{ar} and N_0 respectively the number of photons in the superposed, arbitrary and thermal streams we find the factorial moment of order n of N_s is given by

$$\mathrm{E}[N_s^{(n)}] = \int P(\alpha)|\alpha|^{2n} d^2\alpha \ . \tag{A.3}$$

We next substitute for $P(\alpha)$ from $(A-1)$ to find

$$\begin{aligned}
\mathrm{E}[N_s^{(n)}] &= \frac{1}{\pi N} \int P(\beta) \mathrm{e}^{-\frac{|\alpha-\beta|^2}{N}} \sum_{m=0}^{n} \sum_{k=0}^{n} \binom{n}{m} \binom{n}{k} (\alpha-\beta)^m (\overline{\alpha}-\overline{\beta})^k \beta^{n-m} \overline{\beta}^{n-k} \mathrm{d}^2\beta \mathrm{d}^2\alpha \\
&= \frac{1}{\pi N} \sum_{m=0}^{n} \sum_{k=0}^{n} \int P(\beta) \mathrm{e}^{-\frac{|\alpha|^2}{N}} \beta^{n-m} \overline{\beta}^{n-k} \alpha^m \overline{\alpha}^k \mathrm{d}^2\beta \mathrm{d}^2\alpha \ .
\end{aligned} \tag{A.4}$$

We note that in view of the occurrence of $|\alpha|^2$ in exponential the integral vanishes unless $k = m$. This simplifies the expression on the r.h.s. and we find

$$\begin{aligned}
\mathrm{E}[N_s^{(n)}] &= \frac{1}{\pi N} \sum_{m=0}^{n} \int P(\beta)|\alpha|^{2n}|\beta|^{2(n-m)} [\binom{n}{m}]^2 \mathrm{e}^{-\frac{|\alpha|^2}{N}} \mathrm{d}^2\alpha \mathrm{d}^2\beta \\
&= \sum_{m=0}^{n} [\binom{n}{m}]^2 \mathrm{E}[N_{ar}^{(n-m)}] \mathrm{E}[N_0^{(m)}] \\
&= \sum_{m=0}^{n} [\binom{n}{m}]^2 m! N^m \mathrm{E}[N_{ar}^{(n-m)}]
\end{aligned} \tag{A.5}$$

in agreement with the result obtained by Loudon and Shepherd [20]; Loudon and Shepherd used the operator technique and also made use of the pure random nature of the phase of the amplitude of thermal stream. Next we note that the probability generating function $G(u)$ of the number of photons can be obtained explicitly in CMTF by using the conditional Poisson structure of the distribution. Thus it follows from (2.24) that

$$\begin{aligned}
G(u) &= \sum_{n=0}^{\infty} \int P(\alpha) \mathrm{e}^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} u^n \mathrm{d}^2\alpha \\
&= \int P(\alpha) \mathrm{e}^{-|\alpha|^2(1-u)} \mathrm{d}^2\alpha \ .
\end{aligned} \tag{A.6}$$

It is also possible to derive an alternate formula for the case where the stream is characterized by an arbitrary CMD $f(.)$. By taking the projection on an arbitrary coherent state α

$$\langle\alpha|f^{1/2}\rangle = \tilde{f}^{1/2}(\overline{\alpha})\mathrm{e}^{-\frac{1}{2}|\alpha|^2} \tag{A.7}$$

we can identify the diagonal element of the density matrix

$$\langle\alpha|\rho|\alpha\rangle = |\langle\alpha|f^{1/2}\rangle|^2 \; . \tag{A.8}$$

From the above relation we can easily show

$$G(u) = \frac{1}{u\pi}\int\int \tilde{f}^{1/2}(\overline{\alpha})\overline{\tilde{f}^{1/2}(\overline{\alpha})}\mathrm{e}^{-\frac{|\alpha|^2}{u}}\,\mathrm{d}^2\alpha \; . \tag{A.9}$$

Finally we note that the Wigner distribution can be connected to the P-function from the relation (3.25). Using the relation

$$\langle\alpha|\rho|\alpha\rangle = \frac{2}{\pi}\int W(z)\mathrm{e}^{-2|z-\alpha|^2}\mathrm{d}^2 z \tag{A.10}$$

it follows from (A.9)

$$G(u) = \frac{2}{\pi(1+u)}\int W(z)\mathrm{e}^{-2\frac{(1-u)}{(1+u)}|z|^2}\mathrm{d}^2 z \; . \tag{A.11}$$

B Appendix

Our objective is to obtain the CMD of the test particle coordinate process; in other words we wish to remove the conditioning on the CMD and this can be achieved only by averaging over the coordinates of the oscillators of the bath. This is best done in a subtle way; we note that $F(t, t_0)$ is a Gaussian process (in complex measure theoretic sense) and hence all that we need is its variance for arbitrary t or mean square value since the mean value is zero. Now for a single (fixed) oscillator with coordinate $R(t)$, the variance of $F(t, t_0)$ is given by

$$VarF(t,t_0) = \frac{C^2}{\omega^2}\int_{t_0}^{t}\int_{t_0}^{t}\sin\omega(t-u)\sin\omega(t-v)\mathrm{E}[R(u)R(v)]\mathrm{d}u\mathrm{d}v \; . \tag{B.1}$$

To obtain the correlation function we note that the conditional measure density as given by (2.11) with suitable change of parameters will yield the required result. It follows from (2.11)

$$\mathrm{E}[R(v)|R(u)] = R(u)\mathrm{e}^{-\mathrm{i}\Omega(v-u)} \; . \tag{B.2}$$

Thus we have

$$\mathrm{E}[R(v)R(u)] = \mathrm{E}\{[R(u)]^2\}\mathrm{e}^{-\mathrm{i}\Omega(v-u)}, v > u \; . \tag{B.3}$$

Next we note that if the bath oscillator is initially maintained in equilibrium at temperature T then it follows that the CMD $\pi(x, t|\text{equilibrium at } T, t_0)$ is given by

$$\pi(x,t|\text{equilibrium at } T, t_0) = \exp -\frac{M\Omega}{\hbar}\frac{x^2}{[(\coth\frac{\Omega\hbar}{2kT}-1)\mathrm{e}^{-2\mathrm{i}\Omega(t-t_0)}+1]} \; . \tag{B.4}$$

Hence it follows that

$$\mathrm{E}\{[R(u)]^2\} = \frac{1}{2}\frac{\hbar}{M\Omega}\left[(\coth\frac{\Omega\hbar}{2kT} - 1)\mathrm{e}^{-2\mathrm{i}\Omega(t-t_0)} + 1\right] . \tag{B.5}$$

Thus the variance of $F(t, t_0)$ is now given by

$$\mathrm{Var}F(t, t_0) = \frac{C^2\hbar}{\omega^2 M\Omega}\left[(\coth\frac{\Omega\hbar}{2kT} - 1)I_2 + I_1\right] , \tag{B.6}$$

where

$$I_1 = \int_{t_0}^{t}\mathrm{d}u\int_{u}^{t}\sin\omega(t-u)\sin\omega(t-v)\mathrm{e}^{-\mathrm{i}\Omega(v-u)}\mathrm{d}v , \tag{B.7}$$

$$I_2 = \int_{t_0}^{t}\mathrm{d}u\int_{u}^{t}\sin\omega(t-u)\sin\omega(t-v)\mathrm{e}^{-\mathrm{i}\Omega(v-u)-2\mathrm{i}\Omega(u-t_0)}\mathrm{d}v . \tag{B.8}$$

On evaluation of the integrals we have

$$\begin{aligned} I_1 = \frac{1}{\omega^2-\Omega^2}\Bigg\{&\mathrm{i}\frac{\Omega}{2}\left[(t-t_0) - \frac{\sin 2\omega(t-t_0)}{2\omega}\right] - \frac{1}{2}\sin^2\omega(t-t_0) \\ &+ \frac{\omega}{\omega^2-\Omega^2}\left[\mathrm{e}^{-\mathrm{i}\Omega(t-t_0)}\{-\mathrm{i}\Omega\sin\omega(t-t_0) - \omega\cos\omega(t-t_0)\} + \omega\right]\Bigg\} \end{aligned} \tag{B.9}$$

$$I_2 = \frac{1}{2(\omega^2-\Omega^2)^2}[\mathrm{i}\Omega\sin\omega(t-t_0) - \omega\cos\omega(t-t_0) + \omega\mathrm{e}^{-\mathrm{i}\Omega(t-t_0)}]^2 . \tag{B.10}$$

The above results correspond to a typical oscillator of the bath; to obtain the variance corresponding to the bath, we note that we simply make the replacements $C \to C_n, \Omega \to \Omega_n$ and sum over n. Thus we have

$$\mathrm{Var}F(t, t_0)|_{\mathrm{Bath}} = \sum\frac{C_n^2\hbar}{\omega^2 M\Omega_n}\left[\coth\frac{\Omega_n\hbar}{2kT}I_2(n) + I_1(n)\right] \tag{B.11}$$

where $I_1(n)$ and $I_2(n)$ are simply obtained from $(B-9)$ and $(B-10)$ by the replacement $\Omega \to \Omega_n$. Thus the average value of CMD over the bath is now given by

$$\pi_{\mathrm{final}}(x, t|x_0, t_0) = \mathrm{E}[\pi(x, t|x_0, t_0; \mathcal{B})] \tag{B.12}$$

$$\begin{aligned} &= \left(\frac{m\omega}{\pi\hbar(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})}\right)^{1/2}\left(\frac{1}{2\pi\mathcal{A}}\right)^{1/2} \\ &\times\int\exp -\frac{z^2}{2\mathcal{A}}\exp\left\{-\frac{m\omega}{\hbar(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})}[x - x_0\mathrm{e}^{-\mathrm{i}\omega(t-t_0)} - z]^2\right\}\mathrm{d}z . \end{aligned} \tag{B.13}$$

On performing the integration we have

$$\begin{aligned} \pi_{\mathrm{final}}(x, t|x_0, t_0) = &\left\{2\pi[\mathcal{A} + \frac{\hbar}{2m\omega}(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})]\right\}^{-1/2}\times \\ &\times\exp\left\{-[x - x_0\mathrm{e}^{-\mathrm{i}\omega(t-t_0)}]^2/[2\mathcal{A} + \frac{\hbar}{m\omega}(1-\mathrm{e}^{-2\mathrm{i}\omega(t-t_0)})]\right\} \end{aligned} \tag{B.14}$$

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