Homework #4: Topology

1. Let A_λ be subsets of X. $x \in (\bigcup_\lambda A_\lambda)^c \Leftrightarrow x \notin A_\lambda$ for all $\lambda \Leftrightarrow$ $x \in A_\lambda^c$ for all $\lambda \Leftrightarrow x \in \bigcap_\lambda A_\lambda^c$.

2. Let $A \xrightarrow{\varphi} B$, $A \subset A'$. $b \in \varphi[A] \Leftrightarrow \varphi(a) = b$ for some $a \in A \Rightarrow$ $\varphi(a) = b$ for some $a \in A' \Rightarrow b \in \varphi[A'] \Rightarrow \varphi[A] \subset \varphi[A']$.

3. (a) Both $\emptyset = X^c$ and $X = \emptyset^c$ are closed.

   (b) If $C_\lambda \equiv \mathcal{O}_\lambda^c$ are closed, $\bigcap_\lambda C_\lambda = \bigcap_\lambda \mathcal{O}_\lambda^c = (\bigcup_\lambda \mathcal{O}_\lambda)^c$ is closed.

   (c) If $C \equiv \mathcal{O}^c$ and $C' = \mathcal{O}'^c$ are closed, then $C \cup C' = (\mathcal{O} \cap \mathcal{O}')^c$ is closed.

4. If $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ are continuous, $\mathcal{O}_Z$ open in Z, then $(\psi \circ \varphi)^{-1}[\mathcal{O}_Z] = \varphi^{-1}[\psi^{-1}[\mathcal{O}_Z]]$ is open in $X \Rightarrow \psi \circ \varphi$ is continuous.

5. $\emptyset$ and X are open as defined. If $\mathcal{O}_\lambda$ are open, let $x \in \bigcup_\lambda \mathcal{O}_\lambda$ $\Rightarrow x \in \mathcal{O}_{\bar\lambda}$ for some $\bar\lambda \Rightarrow B_x^\epsilon \subset \mathcal{O}_{\bar\lambda} \subset \bigcup_\lambda \mathcal{O}_\lambda$ for some $\epsilon > 0 \Rightarrow$ $\bigcup_\lambda \mathcal{O}_\lambda$ is open. If $\mathcal{O}$ and $\mathcal{O}'$ are open, let $B_x^\epsilon \subset \mathcal{O}$, $B_x^{\epsilon'} \subset \mathcal{O}'$, then $B_x^{\min(\epsilon,\epsilon')} \subset \mathcal{O} \cap \mathcal{O}' \Rightarrow \mathcal{O} \cap \mathcal{O}'$ is open

6. If $\mathcal{O}$ is open in $\mathbb{R}$, $\mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x$ where I_x is the guaranteed interval containing x and a subset of $\mathcal{O}$.

7. Any set $A = \bigcup_{a \in A} \{a\} = \bigcap_{a \in A} \{a\}^c$ which (since $\mathbb{R}$ is Hausdorff) is an intersection of open sets

8. $$\begin{tikzcd} \mathbb{R} & \mathbb{R} \times \mathbb{R} \arrow[l] \arrow[r] & \mathbb{R} \\ & \mathbb{R} \arrow[ul, "\cos"] \arrow[u, "\varphi"] \arrow[ur, "\sin"'] & \end{tikzcd}$$

9. Compose $x \mapsto (x - 1/2)\cdot 2$, $(0,1) \to (-1,-1)$ with

$x \mapsto \dfrac{x}{1-|x|}$, $(-1,1) \to \mathbb{R}$ or use the traditional

$x \mapsto \tan(\pi(x - 1/2))$, $(0,1) \to \mathbb{R}$.

It helps to notice that continuous $\mathbb{R} \xrightarrow{\varphi} \mathbb{R}$ is an isomorphism iff φ is monotonic and unbounded.

10. We have $E_\varphi = \{(x,x') : \varphi(x) = \varphi(x')\}$ and

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[rrr, bend right, "\varphi"'] & X/E_\varphi \arrow[r] & \mathrm{Im}\,\varphi \arrow[r, "\beta"] & Y \end{tikzcd}$$

$$x \longmapsto [x] \longmapsto \varphi(x) \longmapsto \varphi(x)$$

commutes as functions from our discussion of $\underline{\text{set}}$.

First notice that with $X \xrightarrow{f} Y \xrightarrow{g} Z$,

- If Y has the topology induced by f, $g \circ f$ continuous $\Rightarrow$ g cont.
- If Y has the topology induced by g, $g \circ f$ continuous $\Rightarrow$ f cont.

(Apply the contravariant preimage functor to prove this).

Since X/E_φ and $\mathrm{Im}\,\varphi$ have the induced topologies the result follows easily.

11. Suppose that $X \xrightarrow{\varphi} Y$ is monic. $\{1\} \overset{x}{\underset{x'}{\rightrightarrows}} X \xrightarrow{\varphi} Y$, $x : 1 \mapsto x$, $x' : 1 \mapsto x'$ are continuous $\Rightarrow$ φ is 1-1. Suppose $X \xrightarrow{\varphi} Y$ is epic. Let $\underline{2} = \{1, 2\}$ with the indiscrete topology.

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \arrow[r, bend left, "y \mapsto 1"] \arrow[r, bend right, "{y \mapsto \begin{smallmatrix} 1 : \text{if } y \in \mathrm{Im}\,\varphi \\ 2 : \text{if } y \notin \mathrm{Im}\,\varphi \end{smallmatrix}}"'] & \underline{2} \end{tikzcd}$$

commutes $\Rightarrow$ $\mathrm{Im}\,\varphi = Y$.

See Gerch for iso.

12. Let A be a subset of X, Then $\emptyset = \emptyset \cap A$ is open, $A = X \cap A$ is open. If $\mathcal{O}_\lambda \cap A$ are open, then $\bigcup_\lambda (\mathcal{O}_\lambda \cap A) = (\bigcup_\lambda \mathcal{O}_\lambda) \cap A$ is open. If $\mathcal{O} \cap A$ and $\mathcal{O}' \cap A$ are open, then $(\mathcal{O} \cap A) \cap (\mathcal{O}' \cap A) = (\mathcal{O} \cap \mathcal{O}') \cap A$ is open.

13. Let A be a collection of subsets closed under intersection and containing X and $\emptyset$. Let $\mathcal{T}$ be arbitrary unions of elements of A. $\mathcal{T}$ clearly contains both $\emptyset$ and X and is closed under arbitrary unions. If $\bigcup_\lambda A_\lambda$, $\bigcup_\eta B_\eta \in \mathcal{T}$, then $(\bigcup_\lambda A_\lambda) \cap (\bigcup_\eta B_\eta) = \bigcup_\lambda \bigcup_\eta (A_\lambda \cap B_\eta)$ is in $\mathcal{T}$ because A is closed under intersection.

14. Because of problem 13 and the fact that the sets $\alpha^{-1}[\mathcal{O}_X] \cap \beta^{-1}[\mathcal{O}_Y]$ with $\mathcal{O}_X, \mathcal{O}_Y$ open in X, Y are closed under intersection.

15. The sphere is compact (why?) but the plane is not, so they cannot be isomorphic (why?).

16. When X has a finite number of points.

17. Suppose $\mathbb{R} \overset{\varphi}{\cong} \mathbb{R}^2$, $x \in \mathbb{R}$, $\Rightarrow \mathbb{R} - \{x\} \overset{\varphi}{\cong} \mathbb{R}^2 - \{\varphi(x)\}$. $\mathbb{R} - \{x\}$, however is connected, but $\mathbb{R}^2 - \{\varphi(x)\}$ is not $\Rightarrow\Leftarrow$.

18. See Genoch: