Homework #3.

1. We have already shown that if an $n$'th degree ($n \geq 2$) polynomial $p$ has $n$ roots, then $p = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)\,q$ where $q$ is a zero degree polynomial $\Rightarrow\Leftarrow$.

2. Let $\Pi \equiv \varphi_1 \varphi_2 \cdots \varphi_m$. $\operatorname{Ker}\Pi = \operatorname{Ker}\varphi_1 \oplus \mathcal{U}$ where for $u \in \mathcal{U}$, $\varphi_1(u) = 0$ iff $u = 0$. $\varphi_2 \cdots \varphi_m(u) \neq 0 \Rightarrow \Pi(u) \neq 0 \Rightarrow \Pi(K) \neq 0 \Rightarrow\Leftarrow$ $\Rightarrow \mathcal{U} = \operatorname{Ker}\varphi_2\varphi_3\cdots\varphi_m$. $\Rightarrow \operatorname{Ker}\Pi = \operatorname{Ker}\varphi_1 \oplus \cdots \oplus \operatorname{Ker}\varphi_m$.

3. Pythagoras and non-degeneracy are easy. For Cauchy-Schwartz, let $w = av + (w - av)$, $a \equiv \langle w,v\rangle / \langle v,v\rangle$ be the orthogonal decomposition of $w$. $\|w\|^2 \leq \dfrac{|\langle w,v\rangle|^2}{\|v\|^4}\|v\|^2$ $\Rightarrow$ C.S.

↗ using Pythagoras.

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2 + \langle v,w\rangle + \langle w,v\rangle \leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$$

$\Rightarrow$ Triangle follows.

4. $\operatorname{area}(v+w, v+w) = 0 + 0 + \operatorname{area}(v,w) + \operatorname{area}(w,v) = 0$.

5. If $L$ is an isometry, $\langle Lv, Lv\rangle = \lambda\lambda^* \langle v,v\rangle = \langle v,v\rangle$.

6. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

7. $\langle v, Lv\rangle \geq 0 \Rightarrow \lambda\langle v,v\rangle \geq 0 \Rightarrow \lambda \geq 0$.

8. $\langle v, (L+M)v\rangle = \langle v, Lv\rangle + \langle v, Mv\rangle \geq 0$.

9. Given $V \xrightarrow{\varphi} V$, let $\varphi_t(v_1, v_2, \ldots, v_m) \equiv \sum_{i=1}^{m} v_1 \wedge v_2 \wedge \cdots \wedge \varphi(v_i) \wedge \cdots \wedge v_m$

$$\begin{tikzcd}
\overbrace{V \times V \times \cdots \times V}^{m} \arrow[r] \arrow[dr, "\varphi_t"'] & \Lambda_m V \arrow[d, "\varphi_t"] \\
& \Lambda_m V
\end{tikzcd}$$

Because $\varphi_t$ is antisymmetric, it extends to a unique $\varphi_t : \Lambda_m V \to \Lambda_m V$.

$\varphi_t(v_1 \wedge v_2 \wedge \cdots \wedge v_m) \equiv \operatorname{trace}\varphi\,(v_1 \wedge v_2 \wedge \cdots \wedge v_m)$.

The usual properties of trace follow.

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