Problem set 2

1. $0 \cdot v = (1-1) \cdot v = v - v = 0.$

2. Let $\{\} \xrightarrow{\phi_A} A$ be the unique function from the empty set to A.

$$\begin{tikzcd} \{\} \arrow[r, "\phi_V"] \arrow[dr, "\phi_W"'] & V = \{\phi_F\} \cong \{0\} \arrow[d, "\gamma"] \\ & W \end{tikzcd} \qquad \gamma : 0 \mapsto 0$$

The diagram indeed commutes for a unique γ, as promised.

5. Suppose $V \xrightarrow{\varphi} W$ is monic and $S \subset V$ is independent. If $a_1\varphi(s_1) + a_2\varphi(s_2) + \dots + a_n\varphi(s_n) = 0 \Rightarrow \varphi(a_1 s_1 + \dots + a_n s_n) = 0$ $\Rightarrow a_i = 0 \Rightarrow \varphi[S]$ is independent. Conversely, all $\{v\}$ are independent sets unless $v=0$, so if φ preserves independence, it may only map 0 to 0.

Suppose $V \xrightarrow{\varphi} W$ is epic and S spans V. Any $w \in W$ satisfies $\varphi(v) = w$ for some $v \in V \Rightarrow w = a_i \varphi(s_i)$ for some a_i, $s_i \in S$. $\Rightarrow \varphi[S]$ spans W. Conversely, suppose φ preserves span. Since V spans itself $\varphi[V]$ spans $W \Rightarrow \varphi$ is epic.

We have already shown that φ is iso iff it is monic & epic.

6. Let $V \xrightarrow{\varphi} W$. Because $V \cong \operatorname{Ker}\varphi \oplus V/\operatorname{Ker}\varphi$, $\dim(V/\operatorname{Ker}\varphi) = \dim(V) - \dim(\operatorname{Ker}\varphi)$.

$$V \longrightarrow V/\operatorname{Ker}\varphi \xrightarrow{\sim} \operatorname{Im}\varphi \longrightarrow W$$

$\Rightarrow \dim(\operatorname{Im}\varphi) = \dim V - \dim(\operatorname{Ker}\varphi).$

7. Let $S \to V$ and $T \to W$, $V \otimes W$ is the free vector space on $S \times T \Rightarrow \dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

8. Let $\mathbb{R}^\infty \equiv \{(r_1, r_2, r_3, \dots) : r_1, r_2, \dots \in \mathbb{R}\}$

$\cong \{(r_1, 0, r_2, 0, r_3, 0, \dots) : r_1, r_2, \dots \in \mathbb{R}\}$.

9. If

$$\begin{tikzcd} F \arrow[r, "\beta"] \arrow[d, "\bar{\mu}"'] & F/A \arrow[dl, bend right, "\bar{\bar{\mu}}_1"'] \arrow[dl, bend left, "\bar{\bar{\mu}}_2"] \\ Z & \end{tikzcd}$$

commuted for $\bar{\bar{\mu}}_1, \bar{\bar{\mu}}_2$, $\bar{\bar{\mu}}_1 = \bar{\bar{\mu}}_2$ because β is epi.

10. Introduce a basis.

11. $V \oplus (U \oplus W) \cong (V \oplus U) \oplus W$

$(v, (u, w)) \mapsto ((v, u), w)$ is a linear map with

Kernel $\{(0, (0, 0))\}$.