Homework # 1. S.Y. June 10, 2004

3. No, because 0 has no inverse.

$$\begin{tikzcd}(\mathbb{R},+) \arrow[r, bend left, "\exp"] & (\mathbb{R}^{>0},\cdot) \arrow[l, bend left, "\log"]\end{tikzcd}$$

$$\exp(a+b) = \exp(a)\exp(b)$$
$$\log(a\cdot b) = \log(a) + \log(b)$$

$\Rightarrow (\mathbb{R},+) \cong (\mathbb{R}^{>0},\cdot)$.

4. $\mathbb{Z}$ mod n has n elements.

5. No, e.g. the union of the x and y axis in $(\mathbb{R}^2,+)$.

6. Not likely since Perm(G) is so huge. To prove this, let $f: G \to \text{Perm}\, G$ be

$f: g \mapsto$ the permutation that swaps g and e

f is clearly 1-1. Suppose that G has two or more non-identity elements g, g'. But then Im f does not contain swap$(g,g') \in \text{Perm}\, G \Rightarrow G \not\cong \text{Perm}\, G$. If $|G| = 1$, or 2, however, you can check that $G \cong \text{Perm}\, G$.

7. No, it's a "semigroup" only since inverses are lacking.

8. Let ~~G~~ $S \to G$ be the free group on set S, $[G,G]$ be $\{gg'g^{-1}g'^{-1} : g, g' \in G\}$ the (normal) commutator subgroup of ~~G~~ G, let A be an abelian group.

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & G \arrow[r, "\beta"] \arrow[d, "\gamma"] & G/[G,G] \arrow[dl, "\gamma'"] \\ & A & \end{tikzcd}$$

$S \xrightarrow{\beta\circ\alpha} G/[G,G]$ is the free abelian group on S.

9. $\mathbb{R}$ is abelian, so all subgroups like $\mathbb{Z}$ are normal.

Consider

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\alpha"] \arrow[dr, "\text{fraction}"'] & \mathbb{R}/\mathbb{Z} \arrow[d, "\gamma"] \\ & (-1,1) \end{tikzcd} \qquad \alpha(r) \equiv r + \mathbb{Z}$$

reals mod 1 addition

$\text{fraction}(r+s) = \text{fraction}(r) + \text{fraction}(s) \mod 1$

is a group homomorphism with $\mathbb{Z} = \text{Ker}(\text{fraction})$.

$\Rightarrow$ there is a unique mono-morphism γ causing the diagram to commute. fraction is onto $\Rightarrow$ γ is epi as well.

$\Rightarrow \mathbb{R}/\mathbb{Z} \cong (-1,1)$.

10. Let H be a normal subgroup of G. Notice that the set of left cosets $\{gH : g \in G\}$ and right cosets $\{Hg : g \in G\}$ are isomorphic as sets (why?). Thus, if H has 2 left cosets, it also has 2 right cosets $\{H, G-H\}$ in both cases. $\Rightarrow$ H is normal.

11. For $\varphi \in \text{Aut}\, G$, we must show that $\varphi \circ C_g \circ \varphi^{-1}$ is also a conjugation. $\varphi \circ C_g \circ \varphi^{-1}(x) = \varphi(g^{-1}\varphi^{-1}(x)\, g)$

$= \varphi(g^{-1})\, x\, \varphi(g) = C_{\varphi(g)}(x)$.

13. If $\text{Ker}\,\varphi$ contains non-identity g, then $\varphi(g) = \varphi(e) = e$ and φ is not 1-1. Conversely, if $\text{Ker}\,\varphi = \{e\}$, and $\varphi(g) = \varphi(g')$, then $\varphi(g g'^{-1}) = e \Rightarrow g g'^{-1} = e \Rightarrow g = g' \Rightarrow \varphi$ is 1-1.

14. Let $\mathbb{R}^{>0}$ be the multiplicative group of positive reals. Note that $\mathbb{R}^* \xrightarrow{abs} \mathbb{R}^{>0}$ is a group homomorphism with $\ker(abs) = \{-1, +1\}$, so

$$\begin{tikzcd}
\mathbb{R}^* \arrow[r] \arrow[dr, "abs"'] & \mathbb{R}^* / \ker(abs) \arrow[d, "\gamma"] \\
& \mathbb{R}^{>0}
\end{tikzcd} \qquad \Rightarrow \mathbb{R}^* / \{-1, +1\} \cong \mathbb{R}^{>0}$$

15. Suppose G, H are groups. Let $G \cup_d H \xrightarrow{\lambda} F$ be the free group on $G \cup_d H$ as a set.

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d, "i_G"'] & G \cup_d H \arrow[d, "\lambda"'] \arrow[dd, dashed, bend left, "\gamma"] & H \arrow[l, "\beta"'] \arrow[d, "i_H"] \\
G \arrow[r, "\lambda \circ \alpha"] \arrow[dr, bend right, "\varphi"'] & F \arrow[d, "\gamma'"'] & H \arrow[l, "\lambda \circ \beta"'] \arrow[dl, bend left, "\psi"] \\
& Z &
\end{tikzcd}$$

commutes

Given any φ, ψ, Z, the set theory direct sum means that the outer rim commutes with some function γ. The freeness of F then guarantees a unique group homomorphism γ' such that

$$\begin{tikzcd}
G \cup_d H \arrow[r, "\lambda"] \arrow[dr, "\gamma"'] & F \arrow[d, "\gamma'"] \\
& Z
\end{tikzcd}$$

commutes. Thus

$$\gamma' \circ \lambda = \gamma, \quad \gamma \circ \alpha = \varphi, \quad \gamma \circ \beta = \psi$$

$$\Rightarrow \gamma' \circ (\lambda \circ \alpha) = \varphi \qquad \gamma' \circ (\lambda \circ \beta) = \psi$$

$\Rightarrow (F, \lambda \circ \alpha, \lambda \circ \beta)$ is the direct sum of G and H.