Mathematical Physics
Saul Youssef, 2004

# Metric Spaces

A metric space is a set $X$ with a distance $d : X \times X \to \mathbb{R}^{\geq 0}$ such that

$$d(x,z) \leq d(x,y) + d(y,z) \quad \text{triangle inequality}$$

$$d(x,y) = d(y,x)$$

$$d(x,y) = 0 \iff x = y$$

We'll always assume that $X$ is given the standard topology generated by open balls.

def: A sequence $x_0, x_1, x_2, \ldots$ in $X$ is a Cauchy sequence if, for any $\epsilon > 0$, there is an $N$ such that $d(x_i, x_j) < \epsilon$ for $i,j > N$.

def: A metric space is complete if every Cauchy sequence converges to a limit.

def: ~~X~~ Continuous $X \xrightarrow{f} Y$ is uniformly continuous if it is continuous and if, given any $\epsilon > 0$, there is a $\delta > 0$ such that $d(x,x') < \delta \Rightarrow d(f(x), f(x')) < \epsilon$.

def: Continuous $X \xrightarrow{f} X$ is a contraction mapping if $d(f(x), f(x')) \leq d(x,x') \cdot K$ for some fixed $0 \leq K < 1$.

Theorem: A continuous function on a compact metric space is uniformly continuous. Proof: Homework.

Theorem: A contraction mapping has a unique fixed point.

Proof. Homework.

# Typical Applications

## Riemann Integration $[a,b] \xrightarrow{f} \mathbb{R}$ continuous

"Partitions" $\pi = r_0, r_1, r_2, \dots, r_n$ $r_0 \equiv a$, $r_n \equiv b$ $r_i \leq r_{i+1}$

ordered by inclusion

$\rho_{(r_0, r_1, \dots)} \equiv \max_i (r_{i+1} - r_i)$ the radius of $\pi = r_0, r_1, \dots r_n$

$I_{(r_0, r_1, \dots)} \equiv \sum_{i=0}^{n-1} (r_{i+1} - r_i) f(r_i)$

$\pi_0, \pi_1, \pi_2, \dots$ $\leftarrow$ sequence with $\rho(\pi_i) \to 0$

$I_{\pi_0}, I_{\pi_1}, \dots$ is a Cauchy sequence in $\mathbb{R}$ $\to$ has a limit

$\equiv$ "$\int_{[a,b]} f(x)\, dx$"

## Evolution Equation $\frac{df}{dt} = F(f(t))$ Picard's theorem

$f_0(t) \equiv f(0)$

$f_1(t) \equiv f(0) + \int_0^t F(f_0(s))\, ds$

$\vdots$

$f_{i+1}(t) \equiv f(0) + \int_0^t F(f_i(s))\, ds$

Cauchy sequence, showing that $f$ has a unique solution with $f(0)$.

## Solving Laplace Equation by smoothing

## Convergence of Markov Processes

... Typically very algorithm friendly.

# Differential Topology

As a warm-up, let's consider calculus in the light of what we have done in the course. What does

$$\text{"} \; df_x(h) \equiv \lim_{\lambda \to 0} \frac{f(x+\lambda h) - f(x)}{\lambda} \; \text{"} \quad \text{mean?}$$

Setting: If $X \xrightarrow{f} Y$, evidently $X$ and $Y$ must be both topological spaces and real finite dimensional vector spaces. Geroch shows that if $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$ are required to be continuous, then $\mathbb{R}^n$ must have the standard topology.

Objects: $\mathbb{R}^n$ — monic, epic, iso as in top

Morphisms: Continuous functions

$$\begin{tikzcd} X & X \times Y \arrow[l] \arrow[r] & Y \\ & Z \arrow[ul, "f"] \arrow[u] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[dr, "f"'] & X \oplus Y \arrow[d] & Y \arrow[l] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

$\Rightarrow$ $z \mapsto (f(z), g(z))$ is continuous iff $f, g$ are both continuous

$\Rightarrow$ $f + g : X \to Y$ is continuous

$f : X \times Y \to Z$, even if $f_x : y \mapsto f(x,y)$ and $f_y : x \mapsto f(x,y)$ are continuous for all $x, y$, it may be that $f$ is not continuous.

## What are limits?

$\lim\limits_{x \to a} f(x) = y \overset{\text{def}}{\iff}$ For any $B_y^\epsilon$, there is a $B_a^\delta$ s.t.

$f[B_a^\delta] \subset B_y^\epsilon$.

$$\iff \quad x \mapsto \begin{cases} f(x) & \text{if } x \neq a \\ y & \text{if } x = a \end{cases} \quad \text{is continuous}$$

This has the advantage that you don't need "$\delta, \epsilon$" arguments any more. e.g.

Limits are unique: Suppose $\lim\limits_{x \to a} f(x) = y$, $\lim\limits_{x \to a} f(x) = y'$

$$\begin{cases} f(x) & \text{if } x \neq a \\ y & \text{if } x = a \end{cases} - \begin{cases} f(x) & \text{if } x \neq a \\ y' & \text{if } x = a \end{cases} = \begin{cases} 0 & \text{if } x \neq a \\ y-y' & \text{if } x = a \end{cases} \quad \text{is continuous.}$$

$\Rightarrow y = y'$.

Proof. If $y \neq y'$, since $Y$ is Hausdorff, choose $\mathcal{O} \ni y-y'$, $\mathcal{O} \not\ni 0$.

$\Rightarrow f^{-1}[\mathcal{O}] = \{a\}$ is both open and closed (because $X$ is Haus.).

$\Rightarrow \Leftarrow$. //

Fact. If $\lim\limits_{x \to a} f(x) = y$, $\lim\limits_{x \to a} g(x) = z$, then $\lim\limits_{x \to a} (f+g)(x) = y + z$.

Proof:

$$\begin{cases} f(x) & \text{if } x \neq a \\ y & \text{if } x = a \end{cases} + \begin{cases} g(x) & \text{if } x \neq a \\ z & \text{if } x = a \end{cases} = \begin{cases} f(x)+g(x) & \text{if } x \neq a \\ y+z & \text{if } x = a \end{cases}$$

is continuous. //

Fact: If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are continuous and $\lim_{x\to a} f(x) = y$,

then $\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right)$.

Proof. $g \circ \begin{cases} f(x) & \text{if } x \neq a \\ y & \text{if } x = a \end{cases} = x \mapsto \begin{cases} g(f(x)) & \text{if } x \neq a \\ g(y) & \text{if } x = a \end{cases}$

is continuous. $\Rightarrow \lim_{x\to a} g(f(x)) = g(y)$.

... etc. Let the nice properties of continuous functions do all the work for you.

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Back to $f: X \to Y$

$$df_x(h) \equiv \lim_{\lambda\to 0} \frac{f(x+\lambda h) - f(x)}{\lambda}$$

is now well defined.

$\Leftrightarrow$ $\lambda \mapsto F_x(\lambda, h)$ is continuous for all $h \in X$ where

$$F_x(\lambda, h) \equiv \begin{cases} \dfrac{f(x+\lambda h) - f(x)}{\lambda} & \text{if } \lambda > 0 \\ df_x(h) & \text{if } \lambda = 0 \end{cases}$$

(a) $df_x(a\cdot h) = a\, df_x(h)$.

Proof. $a \neq 0 \Rightarrow$

$$F_x(\lambda/a, a\cdot h) = \begin{cases} \dfrac{f(x+\lambda h) - f(x)}{\lambda / a} & \text{if } \lambda > 0 \\ df_x(a\cdot h) & \text{if } \lambda = 0 \end{cases}$$

$\lambda \mapsto a^{-1}\cdot F_x(\lambda/a, a\cdot h)$ is continuous. $\Rightarrow$ by the uniqueness of limits, $df_x(a\cdot h) = a\cdot df_x(h)$. //

(b) $d_x(f+g) = df_x + dg_x$

Proof. $\lambda \mapsto F_x(\lambda, h) + G_x(\lambda, h)$ is continuous where

$$G_x(\lambda, h) \equiv \begin{cases} \dfrac{g(x+\lambda h) - g(x)}{\lambda} & \text{if } \lambda > 0 \\ dg_x(h) & \text{if } \lambda = 0 . \end{cases} //$$

(c) If $f : X \to \mathbb{R}$ has a minimum at $m \in X$, $df_m = 0$.

Proof:

$$\lambda \mapsto \begin{cases} \dfrac{f(m+\lambda h) - f(m)}{\lambda} & \text{if } \lambda > 0 \end{cases} \geqslant 0$$
$$\qquad\quad \begin{cases} df_m(h) & \text{if } \lambda = 0 \end{cases} \Rightarrow \geqslant 0$$

$\Rightarrow$ $df_m(h) \geqslant 0$ for all $h \in X$. $\Rightarrow$ $df_m = 0$. //

(d) The chain rule: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$G_y(\lambda, k) \equiv \begin{cases} \dfrac{g(y+\lambda k) - g(y)}{\lambda} & \text{if } \lambda > 0 \\ dg_y(k) & \text{if } \lambda = 0 \end{cases}$$

$$\left.\begin{array}{l} \lambda, h \mapsto \lambda, F_x(\lambda, h) \\ \lambda, k \mapsto \lambda, G_y(\lambda, k) \end{array}\right\} \text{ continuous for any } x \in X, y \in Y$$

$\Rightarrow$ $\lambda, h \mapsto G_{f(x)}(\lambda, F_x(\lambda, h))$ is continuous.

$$\Rightarrow \quad \lambda \mapsto \begin{cases} \dfrac{g(f(x) + \lambda F_x(\lambda, h)) - g(f(x))}{\lambda} & \text{if } \lambda > 0 \\ dg_{f(x)}(F_x(\lambda, h)) & \text{if } \lambda = 0 \end{cases}$$

$\Rightarrow$ $d(g \circ f)_x = dg_{f(x)} \circ df_x$. //

The chain rule is the thing that makes the differential into a functor.

$$
\begin{tikzcd}
(X,x) \arrow[r, "f"] \arrow[d, dotted] & (Y,y) \arrow[r, "g"] \arrow[d, dotted] & (Z,z) \arrow[d, dotted] \\
X \arrow[r, "df_x"] \arrow[rr, bend right, "d(g\circ f)_x"'] & Y \arrow[r, "dg_y"] & Z
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{Pointed, real, finite dimensional, topological vector spaces} \arrow[d, dotted] \\
\text{Real finite dimensional vector spaces}
\end{tikzcd}
$$

This is called the "tangent space functor" for reasons which will become apparent soon.

<u>Definitions</u>: Suppose $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}$ is differentiable at $x \in \mathbb{R}^3$. Then $df_x \in (\mathbb{R}^3)^*$. Remember that

$$
\left.\begin{array}{l}
dx : (a,b,c) \mapsto a \\
dy : (a,b,c) \mapsto b \\
dz : (a,b,c) \mapsto c
\end{array}\right\} \text{ is a basis of } (\mathbb{R}^3)^*
$$

$$\Rightarrow \quad df_x \equiv \left(\frac{\partial f}{\partial x}\right)_x dx + \left(\frac{\partial f}{\partial y}\right)_x dy + \left(\frac{\partial f}{\partial z}\right)_x dz$$

defines <u>partial derivatives</u>.

★ The Inverse function theorem:

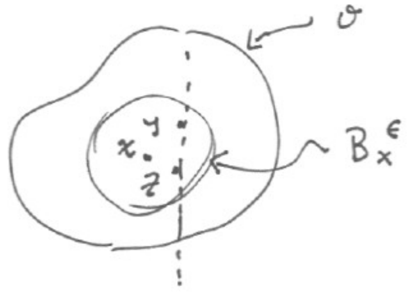
Smooth $X \xrightarrow{f} Y$ is a local diffeomorphism at $x \in X$ iff $df_x$ is an isomorphism.

This is a "deep theorem" according to Guillemin & Pollack, and it certainly is crucial in the theory of manifolds. The proofs that I have seen are all pretty complicated, but at least let me do:

<u>Lemma</u>: If smooth $X \xrightarrow{f} Y$ has $df_x$ iso, then $f$ is locally a set isomorphism.

<u>Proof.</u> Since $x \mapsto \det(df_x)$ is continuous (because $f$ is smooth and det is a polynomial), $df_x$ is iso on some open $\mathcal{O} \ni x$.

Suppose that $f$ fails to be 1-1 on every $B_x^\epsilon \subset \mathcal{O}$. $\Rightarrow$ for each $B_x^\epsilon$, $f(y) = f(z)$ for some $y, z \in B_x^\epsilon$, $y \neq z$.




$\Rightarrow f$ has a maximum $m$ on the line $z \to y$.

$\Rightarrow df_m(y - z) = 0$

$\Rightarrow df_m$ is not iso $\Rightarrow \Leftarrow$.

$\Rightarrow f$ is a local bijective function.

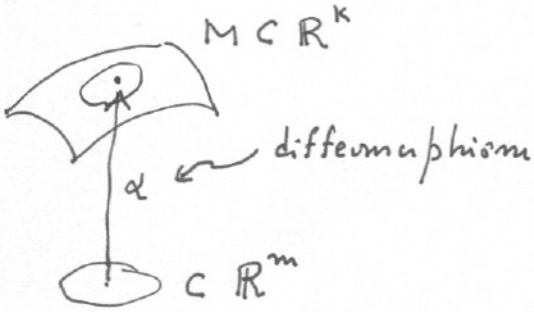
**def**: A function $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$ is *smooth* if it has continuous partial derivatives of all orders.

**def**: Same for $\mathcal{O} \xrightarrow{f} \mathbb{R}^n$.

**def**: If $A$ is a subset of $\mathbb{R}^m$, $A \xrightarrow{f} \mathbb{R}^n$ is *smooth* if, for each $x \in A$, there is an open $\mathcal{O}_x \subset A$ and a smooth map $F : \mathcal{O}_x \to \mathbb{R}^n$ which coincides with $f$ on $A$.

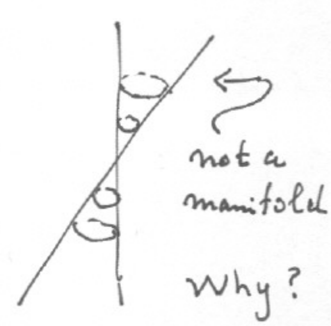
**def**: A map $X \xrightarrow{f} Y$, $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ is a *diffeomorphism* if $f$ is a continuous isomorphism where both $f$ and $f^{-1}$ are smooth.

**def**: A subset $M \subset \mathbb{R}^K$ is an *m-dimensional manifold* if each $x$ in $M$ has an open neighborhood $\mathcal{O}_x \subset M$ which is diffeomorphic to an open subset of $\mathbb{R}^m$



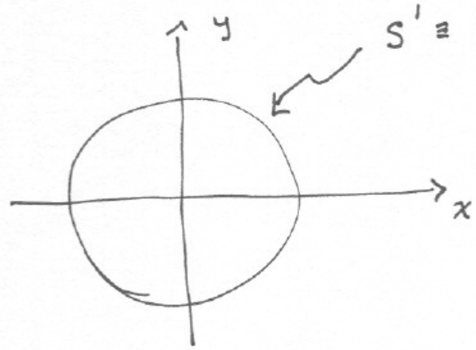

$Aut(\mathbb{R}^N)$

$\mathcal{O}(3)$



Example:

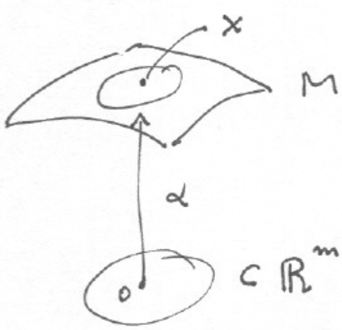
$S^1 \equiv \{ (x,y) : x^2 + y^2 = 1 \} \subset \mathbb{R}^2$




$$\left.\begin{array}{ll} y>0: & (x,y) \mapsto x, \quad x \mapsto (x, \sqrt{1-x^2}) \\ y<0: & (x,y) \mapsto x, \quad x \mapsto (x, -\sqrt{1-x^2}) \\ x>0: & (x,y) \mapsto y, \quad y \mapsto (\sqrt{1-y^2}, y) \\ x<0: & (x,y) \mapsto y, \quad y \mapsto (-\sqrt{1-y^2}, y) \end{array}\right\} \text{ diffeomorphisms.}$$

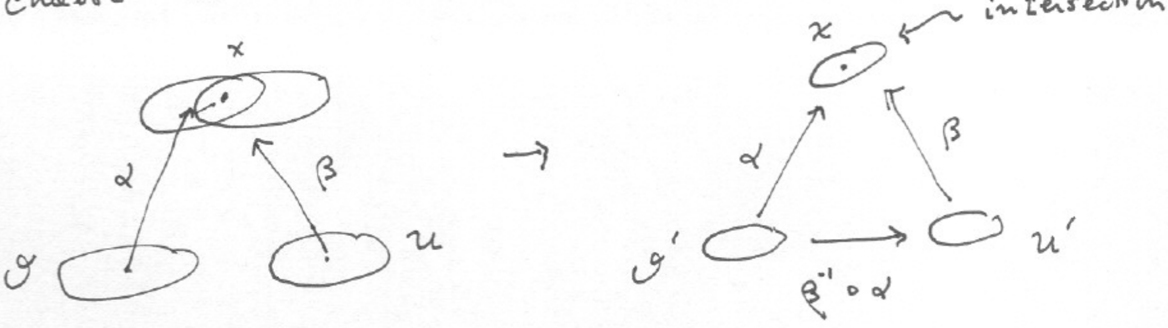
Soon we'll have better ways of constructing manifolds.

## Tangent Spaces




$T_x M \equiv \mathrm{Im}(d\alpha_0)$

This is independent of which diffeomorphism $\alpha$ you choose because




$\Rightarrow \mathrm{Im}\, d\alpha_0 = \mathrm{Im}\, d\beta_0 \equiv T_x M$

As usual, we have a category

Objects: Manifolds

Morphisms: Smooth functions $M \xrightarrow{f} N$

Given $M \xrightarrow{f} N$, $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^\ell$ are manifolds, we wish to define

$$\text{"}df_x\text{"} : TM_x \longrightarrow TN_y \qquad y = f(x).$$

Let $df_x \equiv dF_x|_{TM_x}$ where $F : W \to \mathbb{R}^\ell$ is some function that extends $f$ into $W \subset \mathbb{R}^k$.

$$\begin{tikzcd}
W \arrow[r, "F"] & \mathbb{R}^\ell \\
\mathcal{V} \arrow[u, "\alpha"] \arrow[r, "\beta^{-1}\circ F\circ\alpha"'] & \mathcal{U} \arrow[u, "\beta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{R}^k \arrow[r, "dF_x"] & \mathbb{R}^\ell \\
\mathbb{R}^m \arrow[u, "d\alpha_0"] \arrow[r, "d_0(\beta^{-1}\circ F\circ\alpha)"'] & \mathbb{R}^n \arrow[u, "d\beta_0"']
\end{tikzcd}$$

$\Rightarrow$ $\mathrm{Im}(dF_x) \subset \mathrm{Im}(d\beta_0)$

$\Rightarrow$ $dF_x$ is independent of the choice of extension of $f$.

# The chain rule for manifolds

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, "g"] & P \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto, "f"] & y \arrow[r, mapsto, "g"] & z \end{tikzcd}$$

extend

$$\begin{tikzcd} \mathcal{O}_x \arrow[r, "F"] \arrow[rr, bend right, "G \circ F"'] & \mathcal{O}_y \arrow[r, "G"] & \mathcal{O}_z \end{tikzcd}$$

$$\mathcal{O}_x \subset M \subset \mathbb{R}^k$$
$$\mathcal{O}_y \subset N \subset \mathbb{R}^l$$
$$\mathcal{O}_z \subset P \subset \mathbb{R}^m$$

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "dF_x"] \arrow[rr, bend right, "d(G \circ F)_x"'] & \mathbb{R}^l \arrow[r, "dG_y"] & \mathbb{R}^m \end{tikzcd}$$

$$\begin{tikzcd} TM_x \arrow[r, hook] & \mathbb{R}^k \arrow[r, "dF_x"] \arrow[rrrr, bend right, "d(G \circ F)_x"'] & TN_y \arrow[r, hook] & \mathbb{R}^l \arrow[r, "dG_y"] & TP_z \arrow[r, hook] & \mathbb{R}^m \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} TM_x \arrow[r, "df_x"] \arrow[rr, bend right, "d(g \circ f)_x"'] & TN_y \arrow[r, "dg_y"] & TP_z \end{tikzcd} \quad \text{commutes.}$$

Thus, we have a tangent space functor

$$\begin{tikzcd} (M, x) \arrow[r, "f"] \arrow[d, dashed] & (N, y) \arrow[r, "g"] \arrow[d, dashed] & (P, z) \arrow[d, dashed] \\ TM_x \arrow[r, "df_x"] & TN_y \arrow[r, "dg_y"] & TP_z \end{tikzcd}$$

which gives classification results..

$\Rightarrow$ Manifolds with different dimensions cannot be isomorphic.

<u>The inverse function theorem for manifolds</u>:

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ \mathcal{O} \arrow[u, "\alpha"] \arrow[r, "\bar{f}"] & \mathcal{U} \arrow[u, "\beta"'] \end{tikzcd}$$

$df_x$ iso $\Rightarrow d\bar{f}_0$ iso

$\Rightarrow \bar{f}$ locally iso

$\Rightarrow f$ locally iso.

<u>Regular Values</u>

Given $M \xrightarrow{f} N$,

$x \in M$ is a <u>regular point</u> if $df_x$ is epi

$y \in N$ is a <u>regular value</u> if $f^{-1}[y]$ are all regular points,

$y \in N$ is a <u>critical value</u> if $y$ is not regular.

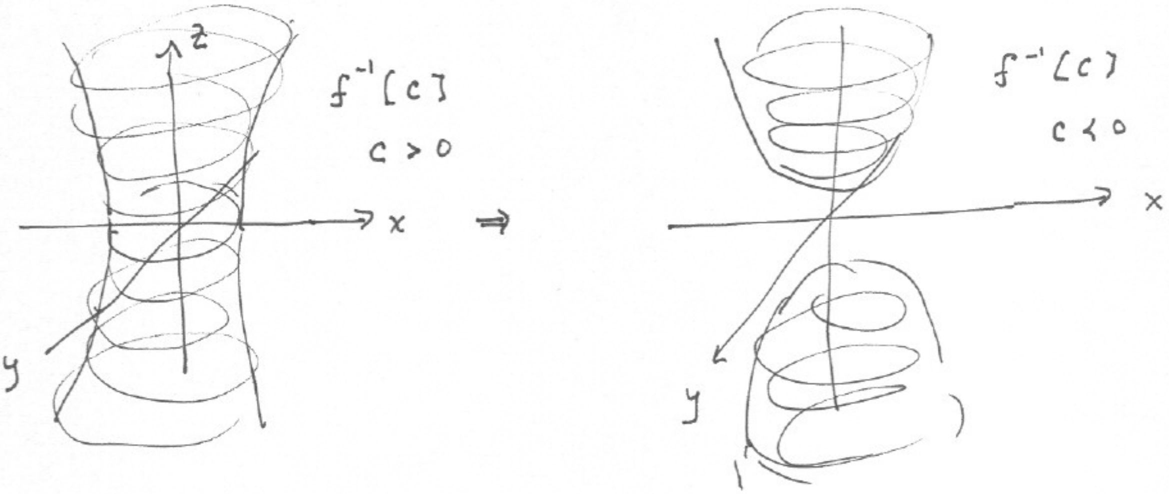
example:

$$f : (x, y, z) \longmapsto x^2 + y^2 - z^2 \qquad \mathbb{R}^3 \to \mathbb{R}^1$$

$$df = 2x\,dx + 2y\,dy - 2z\,dz$$

$\Rightarrow$ $0 \in \mathbb{R}^1$ is the only critical value

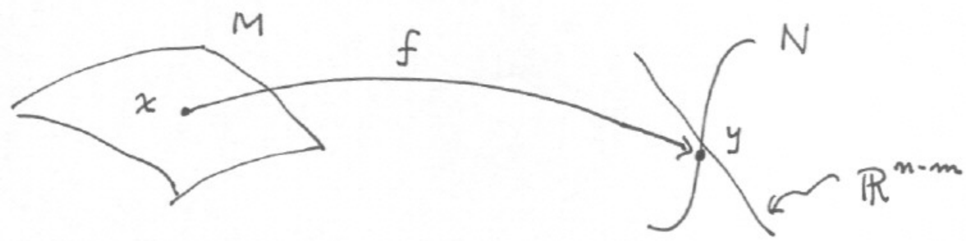
$(0,0,0) \in \mathbb{R}^3$ is the only non-regular point.




$f^{-1}[c]$ changes topology as $c$ goes through it's critical value! $f^{-1}[c]$ is another manifold unless $c = 0$.

Is this generally true? ...

Yes. Suppose $M$ is an $m$-dimensional manifold, $N$ is $n$-dimensional, and smooth $M \xrightarrow{f} N$ ~~is~~ has $df_x$ epic.




$$TM_x \cong \underbrace{\mathrm{Ker}(df_x)}_{n-m} \oplus \underset{m}{\underset{\wr}{\mathcal{U}}}$$

$$F : x \mapsto (f(x), x) \in N \times \mathbb{R}^{n-m} \quad \text{has } dF_x \text{ iso.}$$

By the inverse function theorem, $F$ is locally a diffeomorphism.

$\Rightarrow$ $F : f^{-1}[y] \to y \times \mathbb{R}^{n-m}$ is a local diffeomorphism

$\Rightarrow$ $f^{-1}[y]$ is an $(n-m)$-dimensional submanifold of $M$.

<u>example:</u> $\rho : (x, y, z) \mapsto x^2 + y^2 + z^2$

$\Rightarrow$ $S^2 \equiv \rho^{-1}[1]$ is a manifold.

<u>example:</u> $\mathrm{End}(\mathbb{R}^n)$, the set of $n \times n$ matrices is the manifold $\mathbb{R}^{n^2}$. The subgroup of isometries is given by

$$\psi : L \to L^{+} L$$

$\mathcal{O}(n) = \psi^{-1}[1]$ is a submanifold of $\mathrm{End}(\mathbb{R}^n)$.

This is also a Lie Group.