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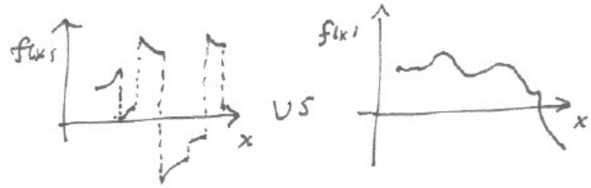
☆ Topology

"nearness"

"continuity"



A <u>topological space</u> is a set X with a collection of subsets identified as "open sets". Open sets must satisfy

- $\emptyset$ and X are open.
- If $\mathcal{O}_\lambda$ are open, so is $\bigcup_\lambda \mathcal{O}_\lambda$.
- If $\mathcal{O}$ and $\mathcal{O}'$ are open, so is $\mathcal{O} \cap \mathcal{O}'$.

<u>Def</u>: $C \subset X$ is <u>closed</u> if it is the complement of an open set.

$\Rightarrow$

- $\emptyset$ and X are closed
- If C_λ are closed, so is $\bigcap_\lambda C_\lambda$.
- If C and C' are closed, so is $C \cup C'$.

<u>Def</u>: A topological space is <u>Hausdorff</u> if, for any $x, y \in X$, $x \neq y$ there are disjoint open sets $\mathcal{O}_x \ni x$, $\mathcal{O}_y \ni y$.

<u>Why these definitions?</u>

Try changing them. You will find that you lose Hausdorffness for, e.g. $\mathbb{R}$ or all subsets become open i.e. the theory collapses.

Examples: Let X be a set.

1). Let all subsets of X be open : "the discrete topology"

2). Let only $\emptyset$ and X be open: "the indiscrete topology"

3). Let $X = \mathbb{R}$ and call a subset $\mathcal{O} \subset \mathbb{R}$ open if it has this property:

- For any $x \in \mathcal{O}$, there is an open ball $B_x^\epsilon \subset \mathcal{O}$, for some $\epsilon > 0$.

$[\, B_x^\epsilon \equiv \{ x' \in X : |x' - x| < \epsilon \}$ for $\epsilon > 0 \,]$.

4.) Let X be a metric space, i.e. a set with $d : X \times X \to \mathbb{R}^{\geq 0}$ satisfying

$$d(x,z) \leq d(x,y) + d(y,z)$$
$$d(x,y) = d(y,x)$$
$$d(x,y) = 0 \quad \text{iff} \quad x = y.$$

Let $\mathcal{O} \subset X$ be open if, for any $x \in \mathcal{O}$, then $B_x^\epsilon \subset \mathcal{O}$ for some $\epsilon > 0$ where $B_x^\epsilon \equiv \{ x' \in X : d(x, x') < \epsilon \}$.

Morphisms

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if the preimage $f^{-1}[\mathcal{O}]$ of any open set $\mathcal{O} \subset Y$ is open in X,

☆ This captures everything one intuitively means by being continuous.

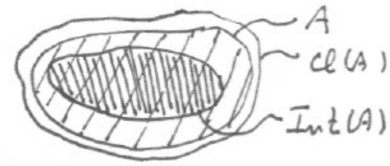
Check that with these morphisms, we have a category called the Category of Topological Spaces.

Characterizing open sets

Theorem: $A \subset X$ is open iff every $x \in A$ is contained in an open subset of A.

Proof: If, for every $x \in A$, $\mathcal{O}_x \subset A$, then $A = \bigcup_{x \in A} \mathcal{O}_x$ is open. //

Let $A \subset X$:



$Int(A) \equiv$ The union of all open subsets of A.

$Cl(A) \equiv$ The intersection of all closed supersets of A.

$Bdy(A) \equiv Cl(A) - Int(A)$

Theorem: (a) $x \in Int(A)$ iff some $\mathcal{O}_x$ is a subset of A.

(b) $x \in Cl(A)$ iff every $\mathcal{O}_x$ intersects A.

(c) $x \in Bdy(A)$ iff every $\mathcal{O}_x$ intersects both A and A^c.

Proof. (a) By definition.

(b) $x \notin Cl(A)$ iff some $\mathcal{O}_x$ is disjoint with A.

(c) Let $x \in Bdy(A)$. If any $\mathcal{O}_x$ fails to intersect A^c,

$\mathcal{O}_x \subset A \Rightarrow x \in Int(A) \Rightarrow \Leftarrow$. //

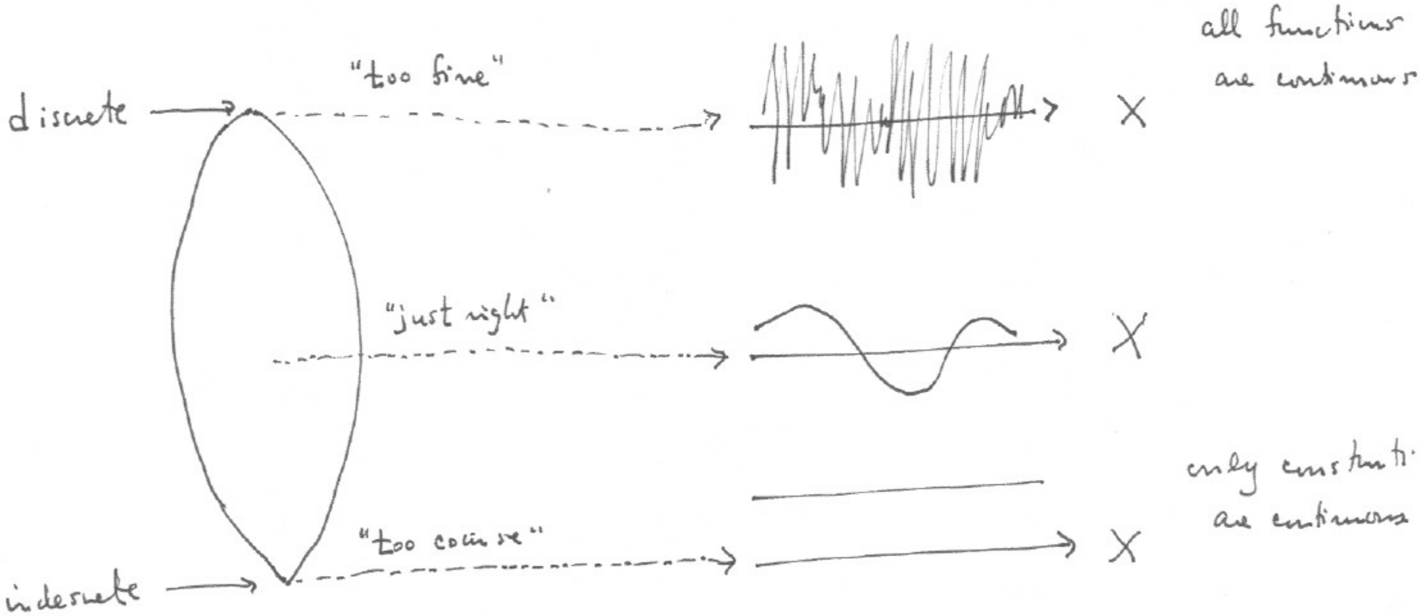
Theorem: Single points in a Hausdorff space are closed.

Proof: Let x be an element of Hausdorff space X. For $x, y \in X$, $x \neq y$, we are guaranteed an $\mathcal{O}_y$ not containing x.

$\Rightarrow X - \{x\} = \bigcup_y \mathcal{O}_y$ is open $\Rightarrow \{x\}$ is closed. //

ex. Let $C([a,b])$ be real valued functions on $[a,b]$, with some topology. The subset of real valued polynomials P is "dense in $C([a,b])$" means $Cl(P) = C([a,b])$. See Weierstrass approximation theorem.

Various topologies on a fixed set



<u>Fact</u>: If $\mathcal{T}, \mathcal{T}' \subset \mathcal{P}(X)$ are topologies, then so is $\mathcal{T} \cap \mathcal{T}'$.

<u>Proof</u>:
- $\emptyset$ and X are both in $\mathcal{T}$.
- If $\mathcal{O}_\lambda$ are in both $\mathcal{T}$ and $\mathcal{T}'$, $\bigcup_\lambda \mathcal{O}_\lambda$ is in $\mathcal{T}$ and $\mathcal{T}' \Rightarrow \bigcup_\lambda \mathcal{O}_\lambda \in \mathcal{T} \cap \mathcal{T}'$
- If $\mathcal{O}, \mathcal{O}'$ are in both $\mathcal{T}$ and $\mathcal{T}'$, $\mathcal{O} \cap \mathcal{O}'$ is in $\mathcal{T}, \mathcal{T}' \Rightarrow \mathcal{O} \cap \mathcal{O}' \in \mathcal{T} \cap \mathcal{T}'$

The same thing works for arbitrary intersections.

$\Rightarrow$ We can define the topology <u>generated by</u> a collection of subsets $A \subset \mathcal{P}(X)$ to be the intersection of all topologies containing A.

<u>Theorem</u>: The real line is generated by the set of open intervals (a, b) with $a, b \in \mathbb{R}$.

<u>Proof</u>. Let $\mathcal{T}$ be the topology generated by open intervals. Since open intervals are open in $\mathbb{R}$, $\mathcal{T} \subset \mathcal{T}(\mathbb{R})$. Also, since every open set in $\mathbb{R}$ is ~~the~~ a union of open balls (intervals), $\mathcal{T}(\mathbb{R}) \subset \mathcal{T}$. //

Topology Induced by a Subset

Let X be a topological space. For any $A \subset X$,

$$\{ A \cap \vartheta : \vartheta \text{ is open in } X \}$$

is a topology on A.

Proof. $A = A \cap X$ and $\emptyset = A \cap \emptyset$ are both open. If $A \cap \vartheta_\lambda$ are open, $\bigcup_\lambda (A \cap \vartheta_\lambda) = A \cap (\bigcup_\lambda \vartheta_\lambda)$ is open. If $A \cap \vartheta$ and $A \cap \vartheta'$ are open, $(A \cap \vartheta) \cap (A \cap \vartheta') = A \cap (\vartheta \cap \vartheta')$ is open. //

Quotient Topologies

Suppose that topological space X has an equivalence relation E defined on it. We can define $\vartheta \subset X/E$ to ~~be~~ be open iff

$$\left(\bigcup_{x \in \vartheta} [x] \right) \text{ is open in } X.$$

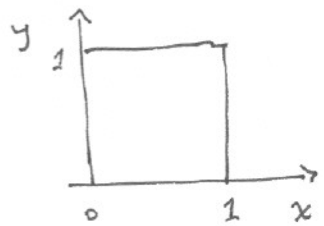
This is the same as the topology induced by the natural map

$$x \longmapsto [x]$$

$$X \longrightarrow X/E$$

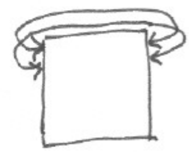
Quotients let us define new spaces by "gluing".

For example, consider the unit square $[0,1] \times [0,1]$ with the topology ~~[illegible]~~ inherited from $\mathbb{R}^2$.

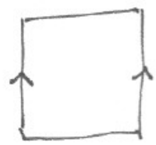


$E \equiv \{((x,y),(x,y)),\ (0,y),(1,y)\}$

identifies



etc. $\equiv$ "



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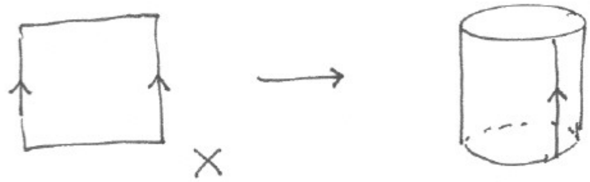
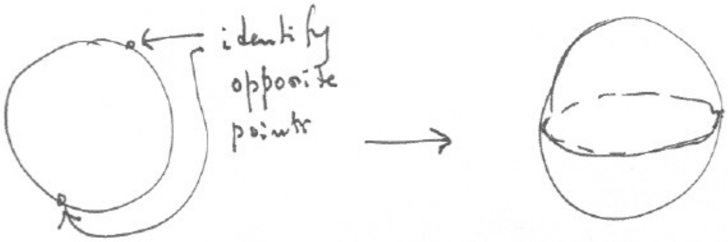


X/E

X/E = Möbius strip

X/E = Torus

a.k.a. "periodic boundary conditions"



$X/E = S^2$ the Sphere

X/E = Klein bottle

Unlike vector spaces, it may be far from obvious whether two topological spaces are isomorphic.

A familiar factorization of any $X \xrightarrow{\varphi} Y$ is available in the category of topological spaces as well. Let

$$E_\varphi = \{ (x, x') : \varphi(x) = \varphi(x') \}$$

be the equivalence relation associated with φ. Then

$$\begin{tikzcd} X \arrow[r, "\text{onto}"] \arrow[rrr, bend right, "\varphi"'] & X/E_\varphi \arrow[r, "\substack{\text{1-1 +} \\ \text{onto}}"] & \operatorname{Im}\varphi \arrow[r, "\text{1-1}"] & Y \end{tikzcd}$$

$$x \longmapsto [x] \longmapsto \varphi(x) \longmapsto \varphi(x)$$

commutes and all of the natural mappings are continuous with X/E_φ having the induced topology and $\operatorname{Im}\varphi$ having the topology inherited as a subset of Y.

Categorical Issues

You can easily check (homework) that continuous $\varphi : X \to Y$ is

- monic iff φ is 1-1
- epic iff φ is onto

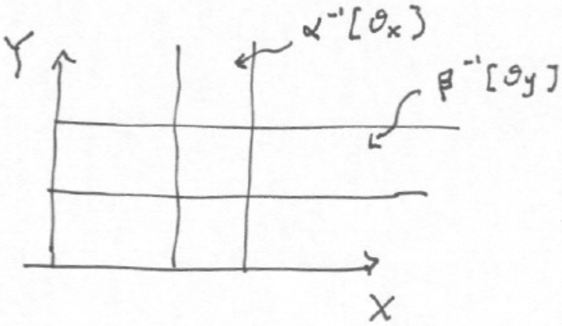
It's also convenient to note that a mapping $\varphi : X \to Y$ is an isomorphism iff both

- φ is 1-1 and onto
- Images and preimages of open sets are open.

Products and sums of topological spaces are also available in this category.

Given topological spaces X, Y, the product is the cartesian product $X \times Y$ with the topology generated by subsets $\alpha^{-1}[\mathcal{O}_X]$, $\beta^{-1}[\mathcal{O}_Y]$ for $\mathcal{O}_X$ open in X and $\mathcal{O}_Y$ open in Y.

$$\begin{tikzcd} X & X \times Y \arrow[l, "\alpha"'] \arrow[r, "\beta"] & Y \\ & Z \arrow[ul, "\varphi"] \arrow[u, "\gamma"] \arrow[ur, "\psi"'] & \end{tikzcd}$$



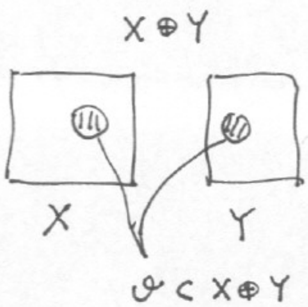
We already know that given φ, ψ continuous, there is a unique γ causing the diagram to commute. The issue is whether γ is continuous (homework).

The sum of X and Y is a topology on $X \oplus Y \equiv X \cup_d Y$

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & X \oplus Y \arrow[d, "\gamma"] & Y \arrow[l, "\beta"'] \arrow[dl, "\psi"] \\ & Z & \end{tikzcd}$$

$\mathcal{O} \subset X \oplus Y$ is open iff both $\alpha^{-1}[\mathcal{O}]$ and $\beta^{-1}[\mathcal{O}]$ are open.

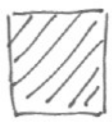
Proof: homework.



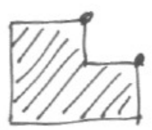
Nets provide an alternative way to think about open sets and continuous maps.

Def. A partially ordered set Δ is a directed set if, for any $\delta, \delta' \in \Delta$, there is a δ'' such that $\delta, \delta' \le \delta''$.

ex:



Directed



Not a directed set

$(x,y) \le (x',y') \iff x \le x'$ and $y \le y'$

ex. The set of open neighborhoods of a point $x \in X$ is a directed set ordered by inclusion.

Def: A net in topological space X is a directed set Δ and a map $\Delta \xrightarrow{\kappa} X$. We often write $x_\delta \equiv \kappa(\delta)$.

Def: A net x_δ in X converges to a limit $x \in X$ if the following condition holds:

Given any $\mathcal{O}_x$, there is a $\delta \in \Delta$ such that $\delta \le \delta' \Rightarrow x_{\delta'}$

Theorem: Limits are unique in a Hausdorff space. ~~Suppose~~

Proof: Suppose that $x_\delta \to x$, $x_\delta \to y$, $x \ne y$. Choose disjoint $\mathcal{O}_x, \mathcal{O}_y$. We are guaranteed a δ_x s.t. $\delta_x \le \delta \Rightarrow x_\delta \in \mathcal{O}_x$, a δ_y s.t. $\delta_y \le \delta \Rightarrow x_\delta \in \mathcal{O}_y$ $\Rightarrow$ let $\delta_x, \delta_y \le \eta$ $x_\eta \in \mathcal{O}_x$, $x_\eta \in \mathcal{O}_y$ $\Rightarrow \Leftarrow$. //

Theorem: Let A be a subset of topological space X. $Cl(A)$ is exactly the set of limit points of nets in A.

Proof: See Geroch. (Note that this does not work for sequences as opposed to nets.)

We can use nets as an alternative way of characterizing continuous functions

<u>Theorem</u>: Let $\varphi : X \to Y$ be a mapping. φ is continuous iff for every net $x_\delta \to x$ in X, the net $\varphi(x_\delta)$ converges to $\varphi(x)$.

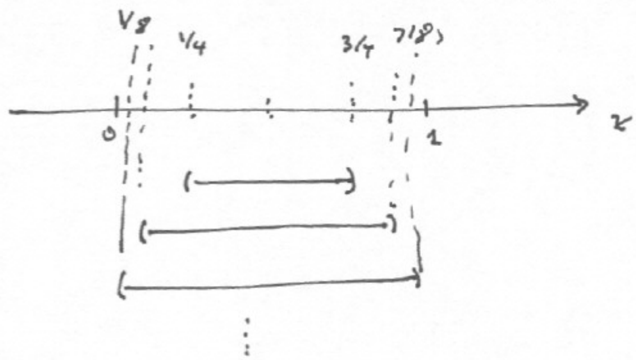
<u>Proof</u>: Suppose that φ is continuous and $x_\delta \to x$ in X. Given any $\mathcal{O}_{\varphi(x)}$, there is an $\mathcal{O}_x$ such that $\varphi[\mathcal{O}_x] \subset \mathcal{O}_{\varphi(x)}$. Let $\delta_x \leq \delta \Rightarrow x_\delta \in \mathcal{O}_x$. $\Rightarrow \delta_x \leq \delta \Rightarrow \varphi(x_\delta) \in \mathcal{O}_{\varphi(x)} \Rightarrow \varphi(x_\delta) \to \varphi(x)$.

Conversely, if φ is not continuous, there must be an $x \in X$ $\mathcal{O}_{\varphi(x)}$ such that $\varphi[\mathcal{O}]$ fails to be in $\mathcal{O}_{\varphi(x)}$ for each open neighborhood $\mathcal{O} \in \Delta_x$ of x ($\Delta_x \equiv$ the directed set of open neighborhoods of x ordered by inclusion). $\Rightarrow$ We can construct a net $\Delta_x \xrightarrow{\eta} X$ by choosing $\eta : \mathcal{O} \mapsto$ some $x \in \mathcal{O}$ s.t. $\varphi(x) \notin \mathcal{O}_{\varphi(x)}$. Then x_η converges to x, but $\varphi(x_\eta)$ fails to converge to $\varphi(x)$.

A topological space is <u>compact</u> if every open cover of the space has a finite subcover.

<u>Compactness</u>

The interval $(0,1) \subset \mathbb{R}$ is not compact, for consider the open covering



These open sets cover $(0,1)$, but any finite number will leave two gaps at 0 and 1.

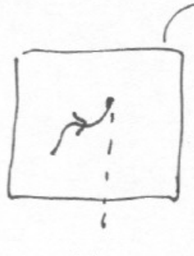
Now we can characterize compact spaces using nets. We say that a net x_δ <u>has an accumulation point</u> at x if, for every $\mathcal{O}_x$, for every $\delta \in \Delta$, there exists $\delta \leq \delta'$ with $x_{\delta'} \in \mathcal{O}_x$.

<u>Theorem</u>: X is compact iff every net in X has an accumulation point.

<u>Proof</u>. Suppose that X is compact but some net x_δ has no accumulation points. $\Rightarrow$ Each $x \in X$ is not an accumulation point for x_δ $\Rightarrow$ for each x, there is an $\mathcal{O}_x$, δ_x s.t. x_δ is outside of $\mathcal{O}_x$ for all $\delta_x \leq \delta$. Let $\{\mathcal{O}_x : x \in A\}$ be a finite subcover. Since A is finite, we can choose $\delta \in \Delta$ s.t. $\delta_x \leq \delta$ for all $x \in A$. $\Rightarrow\Leftarrow$.

See Geroch for the converse...

Example:



X compact topological space

$t \in \mathbb{R}$ "time" is a directed set

$\Rightarrow$ x_t has a point of accumulation.

x_t

Def. A subset A of a topological space is compact if it is compact as a topological space with the induced topology.

Theorem: Any closed subset of a compact space is compact.

Proof. Let C be a closed subset of compact X. For any open cover $\{C \cap \mathcal{O}_\lambda : \lambda \in \Lambda\}$ of C, $\{C \cap \mathcal{O}_\lambda : \lambda \in \Lambda\} \cup \{C^c\}$ is an open cover of X $\Rightarrow$ has a finite subcover $\Rightarrow$ a finite number of $C \cap \mathcal{O}_\lambda$ cover C. //

☆ Theorem: Let $X \xrightarrow{\varphi} Y$ be continuous and C a compact subset of X. Then $\varphi[C]$ is compact.

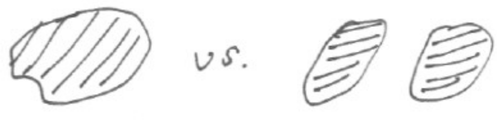
Proof. Let $\mathcal{O}_\lambda$ be an open cover of $\varphi[C]$ $\Rightarrow$ $\varphi^{-1}[\mathcal{O}_\lambda]$ are an open cover of the compact set $\varphi^{-1}[C]$ $\Rightarrow$ some finite number of $\varphi^{-1}[\mathcal{O}_\lambda]$ ~~these~~ cover $\varphi^{-1}[C]$ $\Rightarrow$ $\varphi[\mathcal{O}_\lambda]$ of the same cover ~~so~~ $\varphi[C]$. $\Rightarrow$ $\varphi[C]$ is compact.

☆ example: $x \mapsto 1/x$ on $(0,1) \to \mathbb{R}$ is continuous but unbounded.

But all continuous functions $[0,1] \to \mathbb{R}$ have a maximum and a minimum!

☆ example: $\int_{[a,b]} f(x)\,dx$ exists if f is continuous.

Connectedness



We have the idea that certain topological spaces like $[0,1]$ are different from, say $[0,1] \cup [2,3]$ in that the latter is in two pieces. Can we capture this distinction topologically?

<u>Def.</u> A topological space X is <u>connected</u> if the only subsets of X which are both open and closed are X and $\emptyset$.

This nicely captures the concept:

example: $[0,1]$ is connected, $[0,1] \cup [2,3]$ is not.

example: The real line is connected

example: The rational subset of $\mathbb{R}$ is not connected.

☆ <u>Theorem</u>: Let $X \xrightarrow{\varphi} Y$ be continuous and A a connected subset of X. Then $\varphi[A]$ is connected.

<u>Proof.</u> Suppose $\mathcal{O} \cap \varphi[A] = C \cap \varphi[A]$ is a both-open-and-closed subset of $\varphi[A]$. $\Rightarrow \varphi^{-1}[\mathcal{O}] \cap A = \varphi^{-1}[C] \cap A$ in X $\Rightarrow$ either $\varphi^{-1}[\mathcal{O}] \cap A = A$ ($\Rightarrow \mathcal{O} \cap \varphi[A] = \varphi[A]$) or $\varphi^{-1}[\mathcal{O}] \cap A = \emptyset$ ($\Rightarrow \mathcal{O} \cap \varphi[A] = \emptyset$). //

☆ <u>example</u>: If $X \xrightarrow{\varphi} \mathbb{R}$ and X is both compact and connected, $\varphi[X]$ is both compact and connected, i.e. $\varphi[X]$ is a closed interval $[\varphi_{min}, \varphi_{max}] \subset \mathbb{R}$.

example: Path connected $\Rightarrow$ connected. See Geroch.