Left Overs / Applications / Functors / Algebra   S. Youssef, July 2004

As we were saying last time an inner product on a vector space V induces a natural isomorphism $V \cong V^*$. Let $\langle \cdot, v \rangle$ denote the element of V^* defined by $x \mapsto \langle x, v \rangle$. Let

$$\# : v \mapsto \langle \cdot, v \rangle \,, \quad \# : V \to V^*$$

so that the nondegeneracy of $\langle , \rangle$ is equivalent to $\ker \# = \{0\}$.

$$V \underset{\flat}{\overset{\#}{\rightleftarrows}} V^* \qquad \text{"musical isomorphisms" (finite dim. only.)}$$

These then help to define the "adjoint" of operator $L : V \to V$.

If $L^+ : V \to V$ satisfies

$$\langle Lv, w \rangle = \langle v, L^+ w \rangle \qquad v, w \in V$$

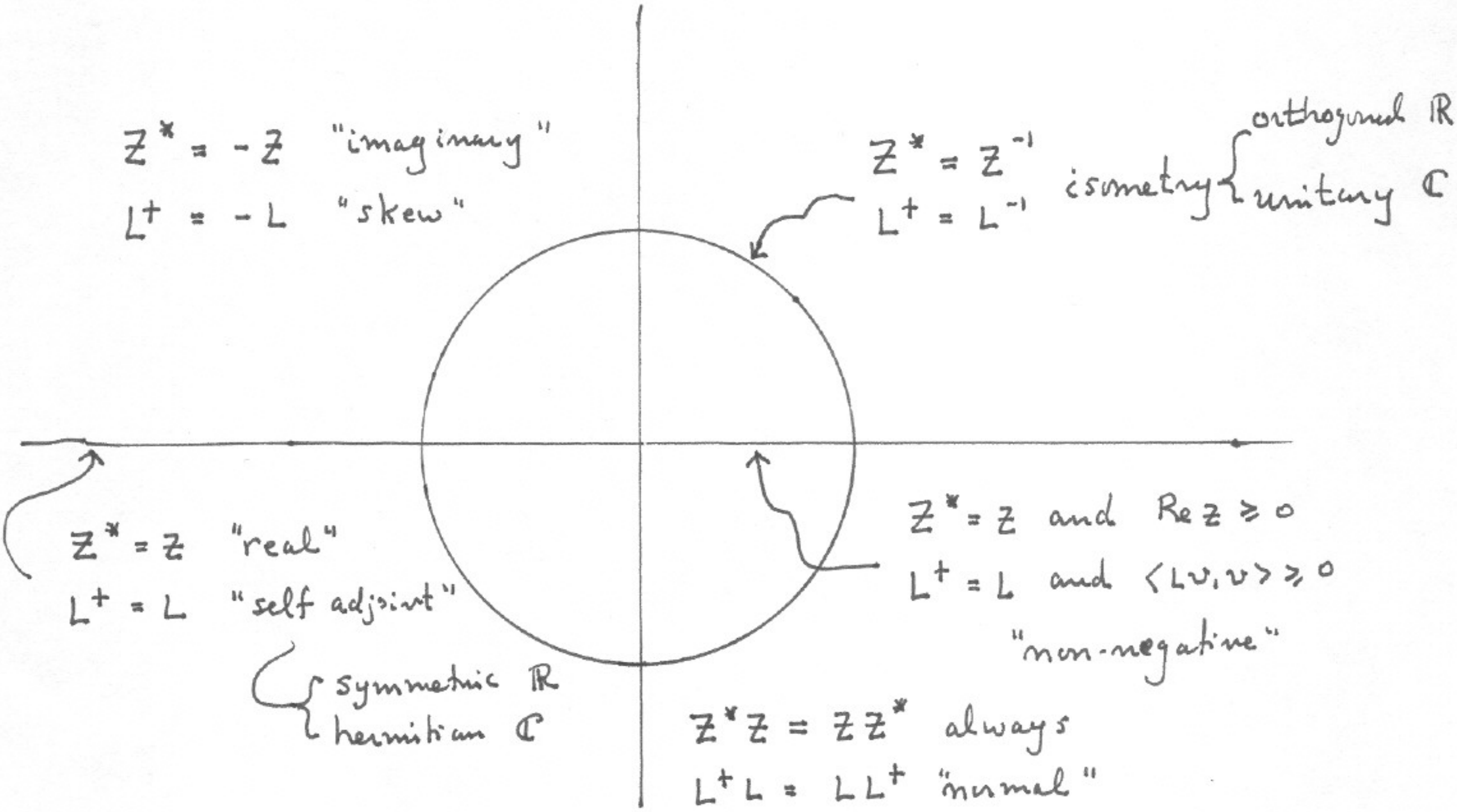
L^+ is called an <u>adjoint</u> of L. Adjoints are unique in finite dimensional spaces because of the following. First note that $(v \mapsto \langle Lv, w \rangle) \in V^*$ and must therefore be equal to $\#(x_w)$ for some $x_w \in V$. $\Rightarrow \#(x_w)(v) = \langle v, x_w \rangle = \langle Lv, w \rangle$ for all $v, w \in V$. $\Rightarrow L^+ w \equiv x_w$ is the unique adjoint of L.

Adjoints have a number of nice and easily proved properties

$(L^+)^+ = L$ (i.e. $+$ is an <u>involution</u>)

$1^+ = 1 \qquad (LM)^+ = M^+ L^+$

L has an inverse iff L^+ has an inverse ....



<u>Complex spectral theorem</u>: V has an orthonormal basis of eigenvectors iff L is normal.

<u>Real spectral theorem</u>: V has an orthonormal basis of eigenvectors iff L is self adjoint.

ex:

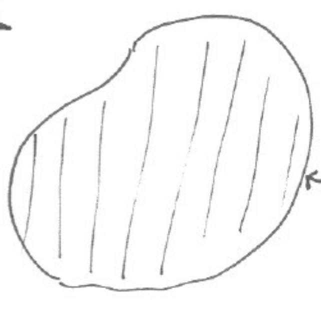
<u>Theorem</u>: Any operator L can be written

$$L = S(L^+L)^{1/2}$$

for some isometry S.

See Axler for proofs.

Application: The Numerical Laplacian



$\Delta\varphi = 0$ Laplace eq.

$\Delta\varphi = \rho$ Poisson eq.

$\partial_t \varphi = \Delta\varphi$ Diffusion eq.

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$\partial_t^2 \varphi = v^2 \Delta\varphi$ Wave eq.

$i\hbar\, \partial_t \psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$ Schrödinger eq.

First let's proceed in the continuous case in hand waving fashion..
This can be done precisely but only later in the course.

We think of Δ as an operator on some vector space $C(\mathcal{U})$ of smooth functions $\mathcal{U} \to \mathbb{R}$, with inner product

$$\langle f, g \rangle \equiv \int_{\mathcal{U}} f(x)\, g(x)\, dx$$

We'll just assume that $\mathcal{U}$ an $C(\mathcal{U})$ are chosen so that this is true.

We'll also assume (Green)

$$\langle \Delta f, g \rangle = \int_{\mathcal{U}} (\Delta f) g = -\int_{\mathcal{U}} (\nabla f)\cdot(\nabla g) + \int_{\partial\mathcal{U}} (\hat{n}\cdot\nabla f)\, g$$

If $\partial\mathcal{U} = \{\}$, Δ is self adjoint, so that if φ is a solution to Laplace, $\langle \Delta\varphi, \varphi\rangle = -\langle \nabla\varphi, \nabla\varphi\rangle = 0 \Rightarrow \nabla\varphi = 0 \Rightarrow \varphi = \text{const.}$ are the only solutions.

e.g. Suppose $\Delta \varphi = \rho$ and f is a solution to Laplace,

$$\Rightarrow \langle \Delta \varphi, f \rangle = \langle \rho, f \rangle = \langle \rho, \Delta f \rangle = 0 \Rightarrow \int_{\mathcal{U}} \rho(x) = 0.$$

e.g. Eigenvalues of Δ are ≤ 0. Suppose $\Delta \varphi = \lambda \varphi$, $\varphi \neq 0$.

$$\langle \Delta \varphi, \varphi \rangle = \lambda \langle \varphi, \varphi \rangle = - \langle \nabla \varphi, \nabla \varphi \rangle \Rightarrow \lambda \leq 0.$$

e.g. $\Delta \varphi = \lambda \varphi$, $\Delta \varphi' = \lambda' \varphi'$ eigenvectors, then

$$\langle \Delta \varphi, \varphi' \rangle = \lambda \langle \varphi, \varphi' \rangle = \lambda' \langle \varphi, \varphi' \rangle$$

$$\Rightarrow \lambda \neq \lambda' \Rightarrow \langle \varphi, \varphi' \rangle = 0.$$

Dropping the boundaryless assumption, we can get variational results: Suppose that we want $\varphi \in C(\mathcal{U})$ to satisfy $\varphi(x) = F(x)$ on $x \in \partial \mathcal{U}$ boundary points. Let

$$\mathcal{A} \equiv \{ \varphi \in C(\mathcal{U}) : \varphi(x) = F(x) \text{ for all } x \in \partial \mathcal{U} \}$$

We claim that if $\varphi \in \mathcal{A}$ minimizes $\mathcal{E}(\varphi) \equiv \langle \nabla \varphi, \nabla \varphi \rangle$, then it is a solution to Laplace with the desired boundary conditions. To see this, let

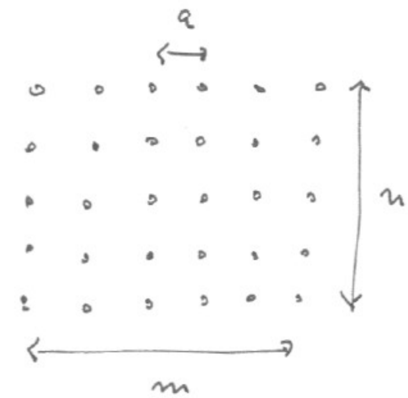
$$\mathcal{A}_0 \equiv \{ \varphi \in C(\mathcal{U}) : \varphi(x) = 0 \text{ for all } x \in \partial \mathcal{U} \}$$

Then if $f \in \mathcal{A}_0$, $\varphi + a \cdot f \in \mathcal{A}$.

$$\mathcal{E}(\varphi + af) = \mathcal{E}(\varphi) + a^2 \mathcal{E}(f) + 2a \langle \nabla \varphi, \nabla f \rangle \quad \text{for all } a \in \mathbb{R}$$

$\Rightarrow \langle \nabla \varphi, \nabla f \rangle = 0 \Rightarrow \langle \Delta \varphi, f \rangle = 0$ for any f. "Weak solution"

$\mathcal{U} \subset \mathbb{R}^2$



$\mathcal{U} = \{0,1,\dots,m-1\} \times \{0,1,\dots,n-1\}$

$C(\mathcal{U})$ is some ∞ dim. real vector space which we have not precisely defined.

$C(\mathcal{U})$ is precisely the free real vector space on the set $\mathcal{U}$.

$$\langle f,g\rangle \equiv \int_{\mathcal{U}} f(x)\,g(x)\,dx$$

$$\langle f,g\rangle \equiv \sum_{u}{}' f_u \cdot g_u$$

We assumed that this was an inner product.

This is an inner product.

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$(\Delta_h f)_{ij} \equiv \{ f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4 f_{ij} \} / a^2$$

Exact Laplacian

Numerical laplacian correct to $\mathcal{O}(a^2)$

Lets look at the numerical Laplacian in more detail.

$$(\Delta_L f)_{ij} \equiv \frac{1}{a^2}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} - 4 f_{ij}\right)$$

can be written in terms of "shift" operators

$$(C_x f)_{ij} \equiv f_{i+1,j} \qquad (C_x^+ f)_{ij} = f_{i-1,j}$$

$$(C_y f)_{ij} \equiv f_{i,j+1} \qquad (C_y^+ f)_{ij} = f_{i,j-1}$$

Obviously, $\langle C_x f, C_x f\rangle = \langle f, f\rangle$, i.e. C_x, C_x^+, C_y, C_y^+ are isometries.

$$\Rightarrow \quad \Delta_L = \frac{1}{a^2}\left(C_x + C_x^+ + C_y + C_y^+ - 4I\right)$$

is self adjoint.

We can also define $D_x \equiv (C_x - I)/a$, $D_y \equiv (C_y - I)/a$

so that

$$-\Delta_L = D_x D_x^+ + D_y D_y^+$$

Now, the whole set of results that we suggested but did not prove in the continuous case follows precisely in the lattice case.

	Continuous	Lattice
	$\mathcal{U} \subset \mathbb{R}^2$	$\mathcal{U} = \{0, 1, 2, \dots, m-1\} \times \{0, 1, \dots, n-1\}$
	$C(\mathcal{U})$ is an ∞-dim. real vector space	$C(\mathcal{U})$ is the $m \times n$ dim. free real vector space on $\mathcal{U}$
	$\langle f, g \rangle \equiv \int_{\mathcal{U}} f(x)\, g(x)\, dx$	$\langle f, g \rangle \equiv \sum'_{u \in \mathcal{U}} f_u \cdot g_u$
	$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$	$-\Delta_L \equiv D_x D_x^+ + D_y D_y^+$
$\partial \mathcal{U} = \emptyset$	$\langle \Delta f, g \rangle = \langle f, \Delta g \rangle$	$\langle \Delta_L f, g \rangle = \langle f, \Delta_L g \rangle$
	$\langle \Delta f, g \rangle = -\langle \nabla f, \nabla g \rangle$	$\langle \Delta_L f, g \rangle = -\langle D_x f, D_x g \rangle - \langle D_y f, D_y g \rangle$
	$\Delta \varphi = 0 \Leftrightarrow \nabla \varphi = 0$	$\Delta_L \varphi = 0 \Leftrightarrow D_x \varphi = D_y \varphi = 0$
	$\Delta \varphi = \rho \Rightarrow \langle \rho, 1 \rangle = 0$	$\Delta_L \varphi = \rho \Rightarrow \langle \rho, 1 \rangle = 0$
	$\mathcal{E}(\varphi) \equiv \langle \nabla \varphi, \nabla \varphi \rangle$	$\mathcal{E}_L(\varphi) \equiv \langle D_x \varphi, D_x \varphi \rangle + \langle D_y \varphi, D_y \varphi \rangle$

Notice that if we had a basis of eigenvectors of C_x, C_y, this would also be a basis of D_x, D_y, D_x^+, D_y^+ and Δ.

Let's find this, but first, switch to $\mathcal{U} = \{0, 1, 2, \dots, n-1\}$

n = even for convenience ...

Since shift is an isometry the only possible eigenvalues are ± 1 (because $\langle C\varphi, C\varphi\rangle = \langle \varphi, \varphi\rangle = \lambda^2 \langle \varphi, \varphi \rangle$), and all eigenvectors are proportional to

$$C \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{or} \quad C \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

and these two certainly don't span $C(\mathcal{U})$ in general.

But don't give up! We can try solving these problems in the free complex vector space on $\mathcal{U}$ instead. Define C as a shift as before, $D \equiv \frac{1}{a}(C - I)$, $\Delta \equiv -DD^{+}$ as before.

$$\langle f, g \rangle \equiv \sum_{u \in \mathcal{U}}{}' f_u g_u^* \quad \text{is the new inner product.}$$

C is still an isometry $\Rightarrow \lambda = e^{i\theta_\lambda}$. Since $C^n = 1$, obviously $\lambda^n = 1$ and the only possible eigenvalues are

$$\lambda \in \{ z^0, z^1, z^2, \dots, z^{n-1} \} \qquad z \equiv e^{2\pi i/n}$$

$$C \begin{pmatrix} z^0 \\ \vdots \\ z^0 \\ z^0 \\ z^0 \end{pmatrix} = z^0 \begin{pmatrix} z^0 \\ \vdots \\ z^0 \\ z^0 \\ z^0 \end{pmatrix} \quad C \begin{pmatrix} z^{n-1} \\ \vdots \\ z^2 \\ z^1 \\ z^0 \end{pmatrix} = z \begin{pmatrix} z^{n-1} \\ \vdots \\ z^2 \\ z^1 \\ z^0 \end{pmatrix} \quad C \begin{pmatrix} z^{2n-1} \\ \vdots \\ z^4 \\ z^2 \\ z^0 \end{pmatrix} = z^2 \begin{pmatrix} z^{2n-1} \\ \vdots \\ z^2 \\ z^0 \end{pmatrix} \quad \dots \quad C \begin{pmatrix} z^{(n-1)(n-1)-1} \\ \vdots \\ z^{n-1} \\ z^0 \end{pmatrix} = z^{n-1} \begin{pmatrix} \\ \end{pmatrix}$$

$\Rightarrow$ We found eigenvectors for all n eigenvalues $\Rightarrow$ this is a basis making the operation of C, D, Δ trivial.

Changing to this basis is called the "Fourier Transform."

[There is a very nice $\mathcal{O}(n \log n)$ FFT algorithm for this].

A functor from category $\mathbb{C}$ to category $\mathbb{C}'$ is a collection of mappings which preserves identities and commuting triangles:

$$\begin{tikzcd}
\mathbb{C} \arrow[d, dashed, "\mathcal{F}"'] & X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] \arrow[d, dashed] & Y \arrow[r, "g"] \arrow[d, dashed] & Z \arrow[d, dashed] \\
\mathbb{C}' & \mathcal{F}X \arrow[r, "\mathcal{F}(f)"] \arrow[rr, bend right, "\mathcal{F}(g \circ f)"'] & \mathcal{F}Y \arrow[r, "\mathcal{F}(g)"] & \mathcal{F}Z
\end{tikzcd}$$

For example:

$$\begin{tikzcd}
\underline{\text{Sets}} \arrow[d, dashed, "\mathcal{P}"'] & X \arrow[r, "f"] \arrow[d, dashed] & Y \arrow[r, "g"] \arrow[d, dashed] & Z \arrow[d, dashed] \\
\underline{\text{Sets}} & \mathcal{P}X \arrow[r, "\mathcal{P}f"] & \mathcal{P}Y \arrow[r, "\mathcal{P}g"] & \mathcal{P}Z
\end{tikzcd}$$

$$\mathcal{P}X \equiv \{\text{the set of subsets of } X\}$$

$$\mathcal{P}f : A \mapsto \{f(a) : a \in A\} \in \mathcal{P}Y$$

$$\mathcal{P}(g \circ f) = \mathcal{P}(g) \circ \mathcal{P}(f)$$

$$\mathcal{P}(i_X) = i_{\mathcal{P}X}$$

This is a functor.

$$\begin{tikzcd}
\mathbb{C} \arrow[d, dashed, "\mathcal{F}"'] & X \arrow[r, "f"] \arrow[d, dashed] & Y \arrow[r, "g"] \arrow[d, dashed] & Z \arrow[d, dashed] \\
\mathbb{C}' \arrow[d, dashed, "\mathcal{G}"'] & \mathcal{F}X \arrow[r, "\mathcal{F}f"] \arrow[d, dashed] & \mathcal{F}Y \arrow[r, "\mathcal{F}g"] \arrow[d, dashed] & \mathcal{F}Z \arrow[d, dashed] \\
\mathbb{C}'' & \mathcal{G}\mathcal{F}X \arrow[r, "\mathcal{G}\mathcal{F}f"] \arrow[rr, bend right, "\mathcal{G}\mathcal{F}(g\circ f)"'] & \mathcal{G}\mathcal{F}Y \arrow[r, "\mathcal{G}\mathcal{F}g"] & \mathcal{G}\mathcal{F}Z
\end{tikzcd}$$

Composition of functors is defined by composing all the corresponding mappings.

- Composition of functors is associative.
- For each category $\mathbb{C}$, there is an identity functor (where all the mappings are identities) which obeys $i_{\mathbb{C}} \circ \mathcal{F} = \mathcal{F}$, $\mathcal{F} \circ i_{\mathbb{C}'} = \mathcal{F}$.
- The category of all categories is a category.

You can also easily verify that

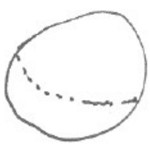
$$\boxed{\text{Functors preserve isomorphisms.}}$$

Pointed topolopological spaces $\overset{\pi}{\dashrightarrow}$ Groups

$$
\begin{tikzcd}
(X,x) \arrow[r, "f"] \arrow[d, dashed] & (Y,y) \arrow[r, "g"] \arrow[d, dashed] & (Z,z) \arrow[d, dashed] \\
\pi(X,x) \arrow[r, "\pi f"] \arrow[rr, bend right, "\pi(g \circ f)"'] & \pi(Y,y) \arrow[r, "\pi g"] & \pi(Z,z)
\end{tikzcd}
$$

$\pi f : [\ell] \mapsto [f \circ \ell]$

"Homotopy"

$\Rightarrow$  $\not\cong$ 

Free constructions are all functors

free vector space on S

$$
\begin{tikzcd}
S \arrow[r] \arrow[d, "f"'] & \mathcal{F}(S) \arrow[d, "\mathcal{F}(f)"] \\
S' \arrow[r] & \mathcal{F}(S')
\end{tikzcd}
$$

$S \cong S' \Rightarrow \mathcal{F}(S) \cong \mathcal{F}(S')$

Vector spaces with isomorphic bases are isomorphic.

Free constructions actually involve two functors

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & V \arrow[d] \\ & W \end{tikzcd}$$

Free : Sets → V.S.s

Forget : V.S.s → Sets

$$\mathrm{Mor}_{\text{Vector Spaces}}(\text{Free } S, W) \cong \mathrm{Mor}_{\text{Set}}(S, \text{Forget } W)$$

These are called "adjoint" functors.

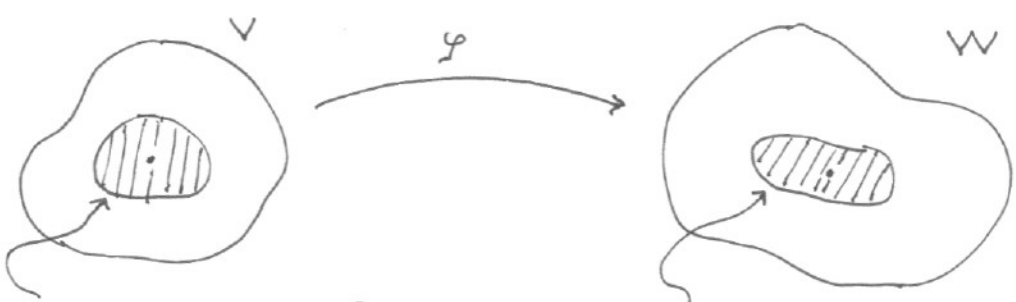
Associative Algebra

An associative algebra is just a vector space V with a bilinear product $V \times V \to V$.

examples:

- Real valued functions on a set with pointwise multiplication.
- $\mathbb{R}^3$ with the usual cross product.
- Operators on a vector space with composition as multiplication.

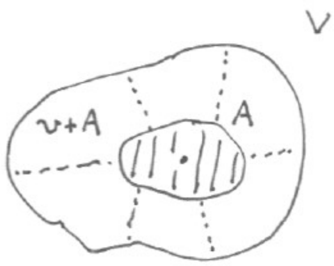
Morphisms: Linear maps $V \xrightarrow{\varphi} W$ with $\varphi(vv') = \varphi(v)\varphi(v')$.



$\mathrm{Ker}\,\varphi \equiv \{ v : \varphi(v) = 0 \}$ $\qquad$ $\mathrm{Im}\,\varphi \equiv \{ \varphi(v) : v \in V \}$

are both subalgebras.

Suppose A is a subalgebra of V



- cosets $v + A$ cover V without overlapping
- cosets are isomorphic as sets
- try to make the cosets into an A.A.

$(v+A) + (v'+A) \mapsto v + v' + A$

$(v+A)\cdot(v'+A) \mapsto vv' + A$

This is a function if

$(v+A, v'+A) = (w+A, w'+A)$

$\Rightarrow vv' + A = ww' + A$

This works if A is an <u>Ideal</u>

def: if $v \cdot a \in A$ $\;a \cdot v \in A$ for all $v \in V$, $a \in A$.

$\Rightarrow v = w + a, \; v' = w' + a' \quad a, a' \in A$

$vv' = ww' + aw' + wa' + aa'$

example $V \equiv \{$ Real functions on $\mathbb{R}$ with pointwise multiplication $\}$

- The subset of bounded functions in V is a subalgebra but not an ideal because a bounded function multiplied by a non-bounded function may not be bounded.
- The subset of functions in V which are zero except on $[0,1]$ is an ideal.

As with groups and normal subgroups, $\mathrm{Ker}\,\varphi$ is an <u>ideal</u> in the category of real or complex a.a.s.

$$\begin{tikzcd}
V \arrow[r, "epi"] \arrow[rrr, bend right, "\varphi"'] & V/\mathrm{Ker}\,\varphi \arrow[r, "iso"] & \mathrm{Im}\,\varphi \arrow[r, "mono"] & W
\end{tikzcd}$$

commutes with natural mappings as usual.

Free associative algebra on V is $V^0 \oplus V^1 \oplus V^2 \oplus \cdots$

$$\begin{tikzcd}
S^0 \oplus S^1 \oplus S^2 \oplus \cdots \oplus \cdots \arrow[r] \arrow[dr, "\varphi"'] & V^0 \oplus V^1 \oplus V^2 \oplus \cdots \arrow[d, "\varphi"] \\
 & W
\end{tikzcd}$$

This is much less scary than it looks. $S^0 \oplus S^1 \oplus \cdots$ is just the set of (possibly empty) lists of elements from the set S and $V^0 \oplus V^1 \oplus \cdots$ are linear combinations of the same with concatenation for multiplication.

Topology

A topological space is a set X with a collection of "open" subsets satisfying

(a) $\emptyset$ and X are both open

(b) If $\mathcal{O}_\lambda$ are open, $\bigcup_\lambda \mathcal{O}_\lambda$ is open

(c) If $\mathcal{O}, \mathcal{O}'$ are open, $\mathcal{O} \cap \mathcal{O}'$ is open

examples:

- Let X be any set. Let open sets be all subsets of X. This is the discrete topology on X.
- Let X be any set. Let only $\emptyset$ and X be open sets. This is the indiscrete topology on X.
- Let X be a metric space and define $B_x^\epsilon = \{ y \in X : d(y,x) < \epsilon \}$. Let a subset $\mathcal{O}$ of X be open if, for each $x \in \mathcal{O}$, there is a $B_x^\epsilon \subset \mathcal{O}$ with $\epsilon > 0$. This is a topology because

(a) Every $x \in \emptyset$ vacuously satisfies the condition.

(b) $B_x^\epsilon \subset X$ for any $\epsilon > 0 \Rightarrow X$ is open.

(c) If $\mathcal{O}_\lambda$ are all open and $x \in \bigcup_\lambda \mathcal{O}_\lambda$, $x \in \mathcal{O}_{\bar\lambda}$ for some $\bar\lambda \Rightarrow B_x^\epsilon \subset \mathcal{O}_{\bar\lambda} \subset \bigcup_\lambda \mathcal{O}_\lambda$ for some $\epsilon > 0$.

(d) If $\mathcal{O}, \mathcal{O}'$ are open, let $x \in \mathcal{O}, \mathcal{O}'$. Let $B_x^\epsilon \subset \mathcal{O}$, $B_x^{\epsilon'} \subset \mathcal{O}' \Rightarrow B_x^{\min(\epsilon,\epsilon')} \subset \mathcal{O} \cap \mathcal{O}' \Rightarrow \mathcal{O} \cap \mathcal{O}'$ is open.

This is the standard topology on $\mathbb{R}^n$.

Def: A set is called "closed" if it is the complement of an open set.

$\Rightarrow$ (a) $\emptyset$ and X are closed

(b) If C_λ are closed, so is $\bigcap_\lambda C_\lambda$

(c) If C and C' are closed, $C \cup C'$ is closed.

Note: Sets may be open, closed, both, neither.

Characterizing open sets

Theorem: A subset A of a topological space X is open iff every $a \in A$ has an open neighborhood $\mathcal{O} \subset A$.

Proof: Suppose A is open. Then $A \subset A$ is certainly an open neighborhood of any $a \in A$. Conversely, suppose that the property holds, then $A = \bigcup_a \mathcal{O}_a$ is open.

Def. Let A be a subset of topological space X.

$\mathrm{Int}(A) \equiv$ The union of all open subsets of A

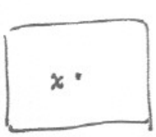
$\mathrm{Cl}(A) \equiv$ The intersection of all closed supersets of A

$\mathrm{Bndy}(A) \equiv \mathrm{Cl}(A) - \mathrm{Int}(A)$

Theorem: A is open iff it contains no boundary points.

Theorem: Single points in a Hausdorff space are closed.

Proof.



$\{x\}^c = \bigcup_y \mathcal{O}_y$ is open

where $x \notin \mathcal{O}_y$, by Hausdorffness.