Vectors I ...... <u>Intermission</u> ...... Vectors II

Mathematical Physics
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What is meant by a "natural mapping"?

$$V \to V^{**} \qquad v \mapsto (f \mapsto f(v))$$

$$V \to V/\mathcal{U} \qquad v \mapsto v + \mathcal{U}$$

this has both an informal and a categorical meaning.

<u>Polynomials</u> over a field F

For our purposes now, you can just think of a polynomial over F as a function $p: F \to F$ which can be written

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

for $a_0, a_1, \dots, a_n \in F$, $a_n \neq 0$. n is called the <u>degree</u> of p.

<u>Fact</u>: If p has a root, λ, $p(x) = (x-\lambda)\,q(x)$ where q is an $n-1$'st degree polynomial.

<u>Proof</u>: Let $p(\lambda)=0$. $d(x) \equiv p(\lambda + x) = a_0 + a_1(x+\lambda) + \dots + a_n(x+\lambda)^n$.

$\Rightarrow d(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$ with $b_n \neq 0$ and $b_0 = 0$ (since $d(0) = p(\lambda) = 0$).

$\Rightarrow d(x) = x\,(b_1 + b_2 x + \dots + b_n x^{n-1})$, $b_n \neq 0$

$\Rightarrow p(x) = d(x-\lambda) = (x-\lambda)\cdot(b_1 + b_2 x + \dots + b_n x^{n-1})$.

<u>Fact</u>: An n'th degree polynomial can have at most n roots.

<u>Proof</u>: (homework).

<u>Theorem</u>: Every nonconstant complex polynomial has a root.

<u>Proof</u>: This is the "fundamental theorem of algebra". We'll prove it only once we get to manifolds.

Let's let "Poly $\leq n$" denote the set of polynomials with degree $\leq n$.

Defining addition and scalar multiplication in the obvious way, we have

<u>Fact</u>: Poly $\leq n$ is an $n+1$-dimensional vector space with basis $1, x, x^2, \dots, x^n$.

<u>Proof.</u> $\{1, x, x^2, \dots, x^n\}$ obviously span Poly $\leq n$. If

$$a_0 \cdot 1 + a_1 \cdot x + a_2 x^2 + \dots + a_n x^n = 0$$

then $a_1 = a_2 = \dots = a_n = 0$ since, otherwise the polynomial would have an infinite number of roots and we just proved that it can have at most n roots. $\Rightarrow a_0 = 0$ also $\Rightarrow$ basis.

Application: Solve $\dfrac{d^3 f}{dx^3} + \dfrac{d^2 f}{dx^2} - f(x) = 0$ $\quad f : \mathbb{R} \to \mathbb{R} \in$

$V \equiv \{$ continuous functions from $\mathbb{R} \to \mathbb{R}$ with continuous derivatives of all orders $\}$

We can actually do better. Let

$$a_n \frac{d^n f}{dx^n} + a_{n-1} \frac{d^{n-1} f}{dx^{n-1}} + \cdots + a_0 f = 0$$

Think of $\frac{d}{dx} \equiv L$ as an operator $V \xrightarrow{L} V$.

$$(a_n L^n + a_{n-1} L^{n-1} + \cdots + a_0 L^0) f \} = 0$$

This can just be factorized like a polynomial

$$a\,(L - \lambda_1)^{m_1} (L - \lambda_2)^{m_2} \cdots (L - \lambda_k)^{m_k} f = 0$$

$\Rightarrow$ The space of solutions is exactly the kernel of

$$\mathrm{Ker}\left[(L-\lambda_1)^{m_1} (L-\lambda_2)^{m_2} \cdots (L-\lambda_k)^{m_k}\right]$$

Since Ker ABC = Ker $A \oplus$ Ker $B \oplus$ Ker C if A, B, C commute.

$$= \mathrm{Ker}\,(L-\lambda_1)^{m_1} \oplus \mathrm{Ker}\,(L-\lambda_2)^{m_2} \oplus \cdots \oplus \mathrm{Ker}\,(L-\lambda_k)^{m_k}$$

You can easily guess that the solution space $\mathrm{Ker}\left(\frac{d}{dx} - \lambda \cdot 1\right)^{m}$ is spanned by $\{ e^{\lambda x}, x e^{\lambda x}, x^2 e^{\lambda x}, \ldots, x^{m-1} e^{\lambda x} \}$.

$\Rightarrow$ Thus, we have found all solutions to

$$a_n \frac{d^n f}{dx^n} + a_{n-1} \frac{d^{n-1} f}{dx^{n-1}} + \cdots + a_0 f = 0 \quad !$$

$\Rightarrow$ Space of solutions is $m_1 + m_2 + \cdots + m_k$ dimensional.

More tensors

We now have quite a few ways to construct vector spaces:

$S \longrightarrow V$ free construction

$V \xrightarrow{\varphi} W$ Ker φ and Im φ

$V \oplus W$ Direct product / sum

V / U Quotient with a subspace

V^* Duals

$\mathrm{Mor}(V, W)$ Maps with pointwise addition

Last time, we added the tensor product $V \otimes W$ to this list:

$$
\begin{tikzcd}
V \times W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[r] \arrow[d, "\bar{\mu}"'] & F/A \equiv V \otimes W \arrow[dl, "\bar{\mu}"'] \\
& Z &
\end{tikzcd}
$$

$\text{"}vw\text{"} \equiv \alpha(v, w)$

$A \equiv$ The vector space generated by

$$
\begin{aligned}
\{ & (v+v')w - vw - v'w, \\
& v(w+w') - vw - vw', \\
& (av)w - a(vw) \\
& v(aw) - a(vw) \}
\end{aligned}
$$

$v, v' \in V$, $w, w' \in W$

$a \in$ the field

This "forces $\bar{\mu}$ to be bilinear."

Let's look at this more explicitly...

$$V \times W \longrightarrow F \longrightarrow V \otimes W$$

$$(v, w) \mapsto a_1 v_1 w_1 + a_2 v_2 w_2 + \dots + a_n v_n w_n \mapsto a_1 v_1 w_1 + a_2 v_2 w_2 + \dots + a_n v_n w_n + A \in F/A$$

an ∞ dim. vector space

$$\equiv V \otimes W$$

ex.

$$vw + vw' + A \quad \text{a coset of } A \text{ in } F$$

$$= v(w+w') - v(w+w') + vw + vw' + A$$

$$= v(w+w') + A$$

Its easy! "A" just absorbs everything that you would cancel by hand if you assumed that the product was bi-linear.

Often, the "A" is dropped and the product is written with $\otimes$ as in

$$v(w+w') + A \equiv v \otimes (w+w')$$

The bottom part of the diagram

$$
\begin{tikzcd}
V \times W \arrow[r] \arrow[dr, "\mu"'] & F \arrow[r] \arrow[d, "\bar{\mu}"] & F/A \equiv V \otimes W \arrow[dl, "\bar{\bar{\mu}}"] \\
 & Z &
\end{tikzcd}
$$

look at these...

$\bar{\mu}(a_1 v_1 w_1 + a_2 v_2 w_2) = a_1 \mu(v_1, w_1) + a_2 \mu(v_2, w_2)$

↑ guaranteed to be linear by the freeness of F.

$\bar{\bar{\mu}}(v \otimes (w + w')) = \bar{\bar{\mu}}(v \otimes w + v \otimes w') =$

$\bar{\mu}(v \otimes w) + \bar{\mu}(v \otimes w') = \mu(v, w) + \mu(v, w')$

guaranteed to be linear and to annihilate A.

If $K \subset V$ and $L \subset W$ are bases, clearly any

$$v \otimes w = \left(\sum_i{}' a_i k_i\right) \otimes \left(\sum_j{}' b_j l_j\right) = \sum_i{}' \sum_j{}' a_i b_j (k_i \otimes l_j)$$

$\Rightarrow$ and any element of $V \otimes W$ can be written as a finite linear combination of $k \otimes l$ with $k \in K$, $l \in L$. $\Rightarrow$ $V \otimes W$ is the free vector space on $K \times L$ as a set.

OK, but what are tensors good for?

ex: You can represent mappings $\varphi \in \mathrm{Map}(V, W)$ as tensors in $V^* \otimes W$

$$(a_1 f_1 \otimes w_1 + a_2 f_2 \otimes w_2 + \dots + a_n f_n \otimes w_n) : v \mapsto$$
$$a_1 f_1(v) w_1 + a_2 f_2(v) w_2 + \dots + a_n f_n(v) w_n$$

In the finite dimensional case, this is an isomorphism $V^* \otimes W \cong \mathrm{Map}(V, W)$ [$\Rightarrow \mathrm{Map}(V,W)$ is an $\dim(V) \cdot \dim(W)$ dimensional vector space, incidentally].

<u>For example:</u> $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

can be written as

$$\cos\theta \; x \otimes f_x - \sin\theta \; y \otimes f_x + \sin\theta \; x \otimes f_y + \cos\theta \; y \otimes f_y \in (\mathbb{R}^2)^* \times \mathbb{R}^2$$

<u>example:</u> $V = \mathbb{R}^4$, the "Lorentz metric tensor" is

$$dx \otimes dx + dy \otimes dy + dz \otimes dz - dt \otimes dt = g \in V^* \otimes V^*$$

sometimes written "$dx^2 + dy^2 + dz^2 - dt^2$" in G.R. books.

<u>example:</u> The <u>quaternions</u> H are usually introduced as a 4-d real vector space with basis $1, i, j, k$ with a bilinear associative product defined by

$$i^2 = j^2 = k^2 = -1 \quad ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j$$

but what does this mean exactly?

Let H be the free vector space on $\{1, i, j, k\}$.

Bilinear products on H are isomorphic to $\mathrm{Mor}(H \otimes H, H)$ via

$$\begin{tikzcd} H \times H \arrow[r] \arrow[dr, "\text{Product}"'] & H \otimes H \arrow[d, "\text{product}"] \\ & H \end{tikzcd}$$

and these are isomorphic to H valued functions on

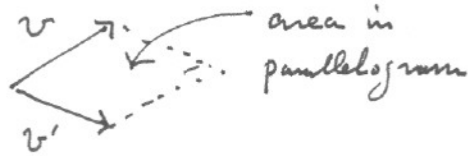
$$\begin{tikzcd} \{1, i, j, k\} \times \{1, i, j, k\} \arrow[r] \arrow[dr, "\times"'] & H \otimes H \arrow[d] \\ & H \end{tikzcd}$$

so $\{\text{bilinear products on } H\} \cong \{4 \times 4 \ H \text{ valued matrices}\}$

$\Rightarrow$ Bilinear products on H form a 64-dimensional vector space. To define the product we have to supply a 4×4 matrix:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Let's try the same line of reasoning for functions having more properties than just being bilinear. For example, consider

area $(v, v') \equiv$ 

$\in \mathbb{R} \quad v, v' \in \mathbb{R}^3 \equiv V$

area is evidently bilinear. Also, however ~~and~~ area$(0, v') =$ area$(v, 0) = 0$
$\Rightarrow$ area$(v, v') = -$area(v', v) it is <u>anti-symmetric</u>.

$$\begin{tikzcd} V \times V \arrow[r] \arrow[dr, "\text{area}"'] & F \arrow[r] \arrow[d, "\overline{\text{area}}"'] & F/B \arrow[dl, "\overline{\overline{\text{area}}}"] \\ & Z & \end{tikzcd}$$

$B \equiv \{$Subspace of F generated by previous A and all elements $vv' + v'v\}$

$F/B \equiv$ "$V \wedge V$" "exterior" or "wedge" product

$$V \times V \longrightarrow F \longrightarrow F/B \equiv V \wedge V$$

(v, v') $\quad a_1 v_1 v_1' + a_2 v_2 v_2' + \dots + a_n v_n v_n'$

$a_1 v_1 \wedge v_1' + a_2 v_2 \wedge v_2' + a_3 v_3 \wedge v_3' + \dots + a_n v_n \wedge v_n'$

$v \wedge v = 0$

$v \wedge v' = -v' \wedge v$ etc.

You can easily see that if $\{x, y, z\}$ is a basis for $V \equiv \mathbb{R}^3$,

$\{x \wedge y, y \wedge z, z \wedge x\}$ is a basis for $V \wedge V$.

These spaces have a nice geometric flavor because of their relationship to area or volumes. Something very nice happens with $\underbrace{V \wedge V \wedge V \wedge \dots \wedge V}_{\dim V \text{ times}} \dots$

Although we've done alot of wedging,

$$\underbrace{V \wedge V \wedge V \wedge \cdots \wedge V}_{n \text{ times}} \qquad \dim V = n$$

$$\text{basis} = \{ \epsilon_1, \epsilon_2, \ldots, \epsilon_n \}$$

is actually one-dimensional! To see this, notice that

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n = \left(\sum_{i=1}^{n} a_i \epsilon_i \right) \wedge \left(\sum_{i=1}^{n} b_i \epsilon_i \right) \wedge \cdots \wedge \left(\sum_{i=1}^{n} z_i \epsilon_i \right)$$

expanded contains only terms with each basis vector occurring exactly once (since $\epsilon_i \wedge \epsilon_i = 0$). Using $\epsilon_i \wedge \epsilon_j = -\epsilon_j \wedge \epsilon_i$, all such terms can be re-arranged in a standard order, or

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n = a\, \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n$$

$\bigwedge_n V$ is one dimensional. Now, let $\varphi \in M_n(V, V)$ and consider

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n \mapsto \varphi(v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n) \equiv K_\varphi\, v_1 \wedge v_2 \wedge \cdots \wedge v_n$$

Since $\bigwedge_n V$ is one dimensional, the action of the map is just multiplication by some constant K_φ. You can easily see that this constant only depends on φ and is independent of which $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ you start with [proof: Any $w_1 \wedge w_2 \wedge \cdots \wedge w_n = a\, v_1 \wedge v_2 \wedge \cdots \wedge v_n$ for some real $a \in \mathbb{R}$.

$\varphi(w_1 \wedge \cdots \wedge w_n) = \varphi(a v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n) = a\, K_\varphi\, v_1 \wedge v_2 \wedge \cdots \wedge v_n$

$= K_\varphi (w_1 \wedge w_2 \wedge \cdots \wedge w_n)$.]

K_φ is called "the determinant" of $\varphi \equiv \det \varphi$

Intuitively, it is clear that $\det \varphi$ is the "volume squashing" factor of φ. We expect, for example, ~~for~~ volume preserving maps like rotations to have $\det \varphi = 1$.

It is also now obvious that $\det i_V = 1$ and

$$\det(\psi \circ \varphi) = \det(\psi) \cdot \det(\varphi) \qquad \varphi, \psi \in \mathrm{Mor}(V, V)$$

making it a group homomorphism to $\mathbb{R}^{\neq 0}$ as we have noted (or $\mathbb{C}^{\neq 0}$ in the complex case).

<u>Theorem</u>: $\varphi \in \mathrm{Mor}(V,V)$ is an isomorphism iff $\det \varphi = 0$.

<u>Proof</u>: If φ is an isomorphism, $\varphi \circ \varphi^{-1} = i_V \Rightarrow \det(\varphi) \cdot \det(\varphi^{-1}) = 1$ $\Rightarrow \det \varphi \neq 0$. Conversely, suppose that φ has some non-zero $v \in V$ in it's kernel. Choose a basis $v, \epsilon_2, \epsilon_3, \dots, \epsilon_n \Rightarrow$

$$\varphi(v \wedge \epsilon_2 \wedge \epsilon_3 \cdots \wedge \epsilon_n) = \varphi(v) \wedge \varphi(\epsilon_2) \wedge \cdots \wedge \varphi(\epsilon_n) = 0 = \det \varphi \cdot v \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n$$

$\Rightarrow \det \varphi = 0$.

<u>Fact</u>. An "eigenvector" of φ is a non zero $v \in V$ s.t. $\varphi(v) = \lambda v$ for $\lambda \in \mathbb{R}$. $\Rightarrow v \in \mathrm{Ker}(\varphi - \lambda \cdot 1) \Rightarrow$ $\det(\varphi - \lambda 1) = 0$.

Isn't that easy?

These volume forms will reappear ~~the~~ as "differential forms" when we do integration on manifolds.

Speaking of Eigenvalues, why are we always looking for them?

$\underbrace{\mathrm{End}(V) \equiv \mathrm{Mor}(V,V)}$ $\dim V = n$

If you are trying to understand $L \in \mathrm{End}\, V$, it helps to know an eigenvector $Lv = \lambda v$, $v \neq 0$ because it gives you an invariant subspace $[v] \equiv \{av : a \in F\}$. You can then split

$$V \cong [v] \oplus \mathcal{U} \quad \text{where } \mathcal{U} \text{ is complementary to } [v].$$

At least for some vectors, then, you know exactly what L does; it just multiplies by λ. The more eigenvectors we know, the more we can split V into one-dimensional subspaces where the action of L is trivial. In the ideal case we might be able to find n different eigenvectors so that V entirely splits up

$$V \cong [v_1] \oplus [v_2] \oplus [v_3] \oplus \ldots \oplus [v_n]$$

so that we have an entire basis of eigenvectors and the action of L is very easy to describe. This is often, but not always achievable [rotation by $90°$ in $\mathbb{R}^2$ has, for example no eigenvectors at all].

This is a subject where the field makes a difference. The complex results are simpler, so we take $F = \mathbb{C}$ for now...

Theorem: Every operator on an n-dimensional complex vector space has an eigenvector.

Proof: Let $L \in \mathrm{End}(V)$ and choose any nonzero v.

If $\dim V = n$, choose $n+1$ vectors like so

$$v,\ Lv,\ L^2 v,\ \dots,\ L^n v$$

Since $\dim V = n$, these vectors must be dependent i.e. there must be $a_0, a_1, \dots, a_n \in \mathbb{C}$ not all zero such that

$$a_0 v + a_1 L v + a_2 L^2 v + \dots + a_n L^n v = 0$$

However, we can factor this like a polynomial $\Rightarrow$

$$a(L-\lambda_1 1)(L-\lambda_2 1)\cdots(L-\lambda_k 1)(v) = 0 \qquad a \neq 0$$

$\Rightarrow$ one of the factors in the chain must zero it's argument

$\Rightarrow (L-\lambda_k \cdot 1)(\bar{v}) = 0$ for some nonzero $\bar{v}$.

Theorem: Eigenvectors with different eigenvalues are independent.

Proof: Suppose $Lv_i = \lambda_i v_i$ with $v_i \neq 0$ and all the λ_i different. $i = 1, \dots, m$. Suppose that $v_1, v_2, \dots, v_m$ are dependent. Let v_k be the first vector spanned by it's predecessors. Then

$$v_k = a_1 v_1 + a_2 v_2 + \dots + a_{k-1} v_{k-1}$$

Applying $(L-\lambda_k 1)$ to both sides,

$$0 = a_1(\lambda_1 - \lambda_k)v_1 + a_2(\lambda_2 - \lambda_k)v_2 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_k$$

$\Rightarrow a_1 = a_2 = \dots = a_{k-1} = 0$ (since the λ_i are all different) $\Rightarrow v_k = 0 \Rightarrow \Leftarrow$.

$\Rightarrow$ the whole list $v_1, v_2, \dots, v_k$ must be independent.

Analogous results for real vector spaces are slightly harder to prove (see Axler's book).

<u>Theorem</u>: Every operator on a finite dimensional nonzero real vector space has an invariant subspace of dimension 1 or 2.

<u>Theorem</u>: Every operator on an odd-dimensional real vector space has an eigenvalue.

This is why, for example "coordinate axes can be 2-dimensional" in Minkowski space (see Geroch's Minkowski space chapter).

Inner Product Spaces

A vector space V with a function $\langle\, ,\rangle : V \times V \to F$ is an <u>inner product space</u> if

$$\langle v + a v', w\rangle = \langle v, w\rangle + a\langle v', w\rangle \qquad \text{linearity in the 1st arg.}$$

$$\langle v, v\rangle \geq 0 \quad \langle v, v\rangle = 0 \Leftrightarrow v = 0 \qquad \text{positivity}$$

$$\langle v, w\rangle = \langle w, v\rangle^*$$

The associated <u>norm</u> is $\|v\| \equiv \langle v, v\rangle^{1/2}$. $\quad \langle v, w\rangle = 0 \Leftrightarrow$ "$v \perp w$"

Warning: Definitions of the term "inner product" vary a bit. By this definition, for example the Minkowski metric is <u>not</u> an inner product.

Fact (Pythagorus): $v \perp w \Rightarrow \|v+w\|^2 = \|v\|^2 + \|w\|^2$.

Fact (Cauchy-Schwartz): $\langle v, w \rangle \le \|v\| \, \|w\|$.

Fact (Non degeneracy): $\langle v, x \rangle = 0$ for all $x \in V \Rightarrow v = 0$.

Fact (Triangle): $\| v + w \| \le \|v\| + \|w\|$

Examples:

$$V = \mathbb{R}^3, \quad \langle v, w \rangle \equiv v_x w_x + v_y w_y + v_z w_z$$

$$V = \mathbb{C}^3, \quad \langle v, w \rangle \equiv v_x w_x^* + v_y w_y^* + v_z w_z^*$$

$$V \equiv \{ \text{continuous real functions on } [0,1] \}$$

$$\langle f, g \rangle \equiv \int_0^1 f(x)\, g(x)\, dx$$

Fact: Gram-Schmidt algorithm

Given a basis $\epsilon_1, \epsilon_2, \ldots, \epsilon_n \in V$, we can construct an orthonormal basis by the well known Gram-Schmidt algorithm

$$\eta_1 \equiv \epsilon_1$$

$$\eta_n \equiv \epsilon_n - \sum_{i=1}^{n-1} \langle \epsilon_n, \eta_i \rangle \eta_i \, / \langle \eta_i, \eta_i \rangle$$

Fact: If $\mathcal{U}$ is a subspace of V, $\mathcal{U} \oplus \mathcal{U}^\perp$ are complementary with

$$\mathcal{U}^\perp \equiv \{ w \in V : \langle u, w \rangle = 0 \text{ for all } u \in \mathcal{U} \}$$

Fact (Parallelogram): $\|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right)$.

Theorem: Suppose $\mathcal{U}$ is a subspace of V and $v \in V$. Then $\|v - \pi_{\mathcal{U}} v\| \le \|v - u\|$ for all $u \in \mathcal{U}$ with equality iff $u = \pi_{\mathcal{U}} v$.

Proof. Suppose $u \in \mathcal{U}$,

$$\|v - \pi_{\mathcal{U}} v\|^2 \le \|v - \pi_{\mathcal{U}} v\|^2 + \|\pi_{\mathcal{U}} v - u\|^2$$

$$= \|(v - \pi_{\mathcal{U}} v) + (\pi_{\mathcal{U}} v - u)\| \quad \text{(Pythagoras)}$$

$$= \|v - u\|^2$$

Problem: Fit a continuous real function f on $[0,1]$ to a 5th degree polynomial.

Solution: Let $V \equiv \{$real continuous functions on $[0,1]\}$ (∞ dimensional) with $\langle f, g\rangle \equiv \int_0^1 f(x)g(x)\,dx$

Let $\mathcal{U} \subset V$ be $\mathrm{Poly}_{\le 5}$.

Let $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5$ be the orthonormal basis of $\mathcal{U}$ gotten from $1, x, x^2, x^3, x^4, x^5$ by Gram-Schmidt.

$\|f - u\|$ is minimized by $u = \pi_{\mathcal{U}} f = \sum_{i=0}^{5} \langle f, \epsilon_i\rangle \epsilon_i$.

This works very well - much better than a Taylor's expansion, for example (typically).

Operators on Inner Product Spaces (finite dimensional)

An inner product on a finite dimensional vector space V fixes a natural isomorphism $V \cong V^*$. To see what this is, denote $x \mapsto \langle v, x \rangle$ by "$\langle v, \cdot \rangle$" $\in V^*$. Let

$$\psi : v \mapsto \langle v, \cdot \rangle, \quad \psi : V \to V^*$$

so that the nondegeneracy of $\langle , \rangle$ is equivalent to $\operatorname{Ker} \psi = \{0\}$.

Given $V \xrightarrow{L} V$, "the" adjoint "L^+" of L is an operator satisfying

$$\langle w, Lv \rangle = \langle L^+ w, v \rangle \qquad v, w \in V$$

We can say "the adjoint" (and promote $+$ to a function) because adjoints are unique. To see this, note that

$$v \mapsto \langle u, Lv \rangle$$

is an element of V^* and must therefore be $\psi(x_u)$ for some unique $x_u \in V$. $\Rightarrow \psi(x_u)(v) = \langle x_u, v \rangle = \langle u, Lv \rangle$ for all $u, v \in V$. Let $L^+ u \equiv x_u$.

Adjoints have a number of nice and easily proved properties:

$(L^+)^+ = L$ ("$+$ is an "involution")

$1^+ = 1$, $(LM)^+ = M^+ L^+$

L has an inverse iff L^+ has an inverse ...

An operator which has $L = L^{\dagger}$ is called <u>self adjoint</u>

$$\text{"self adjoint"} \equiv \begin{cases} \text{"symmetric" in real vector spaces} \\ \text{"hermitian" in complex vector spaces} \end{cases}$$

Self adjoint operators also have nice and easily proved properties.

<u>Fact</u>: Eigenvalues of a self-adjoint operator are real.

<u>Proof</u>: Suppose that $Lv = \lambda v$ is an eigenvector.

$\langle Lv, v\rangle = \lambda \langle v, v\rangle = \langle v, Lv\rangle = \lambda^* \langle v, v\rangle \Rightarrow \lambda = \lambda^*$.

<u>Theorem</u>: Every self-adjoint operator has an eigenvector.

<u>Proof</u>: Axler

<u>Def</u>: Operator L is <u>normal</u> if $L^{\dagger} L = L L^{\dagger}$.

<u>Complex Spectral Theorem</u>: Let L be an operator on a complex finite dimensional vector space. V has an orthonormal basis of eigenvectors of L iff L is normal.

<u>Real Spectral Theorem</u>: Let L be an operator on a real finite dimensional vector space. V has an orthonormal basis of eigenvectors of L iff L is self adjoint.

An operator L is said to be an <u>isometry</u> if it preserves the norm:

$$\|Lv\| = \|v\| \quad \text{for all } v \in V$$

$$\text{isometry} \iff \begin{cases} \text{"orthogonal" in real vector spaces} \\ \text{"unitary" in complex vector spaces} \end{cases}$$

<u>Fact</u>: $\langle Lv, Lw\rangle = \langle v, w\rangle$ for all $v, w \in V$.

<u>Proof</u>: $\langle Lv, Lw\rangle = \frac{1}{4}\left(\|Lv + Lw\| - \|Lv - Lw\|\right)$

$= \frac{1}{4}\left(\|v+w\| - \|v-w\|\right) = \langle v, w\rangle.$

<u>Fact</u>: L has an inverse and $L^{-1} = L^{+}$.

<u>Proof</u>. $\langle (L^{+}L - 1)v, w\rangle = 0$ for all $w \Rightarrow L^{+}L = 1$.

<u>Fact</u>: L is normal.

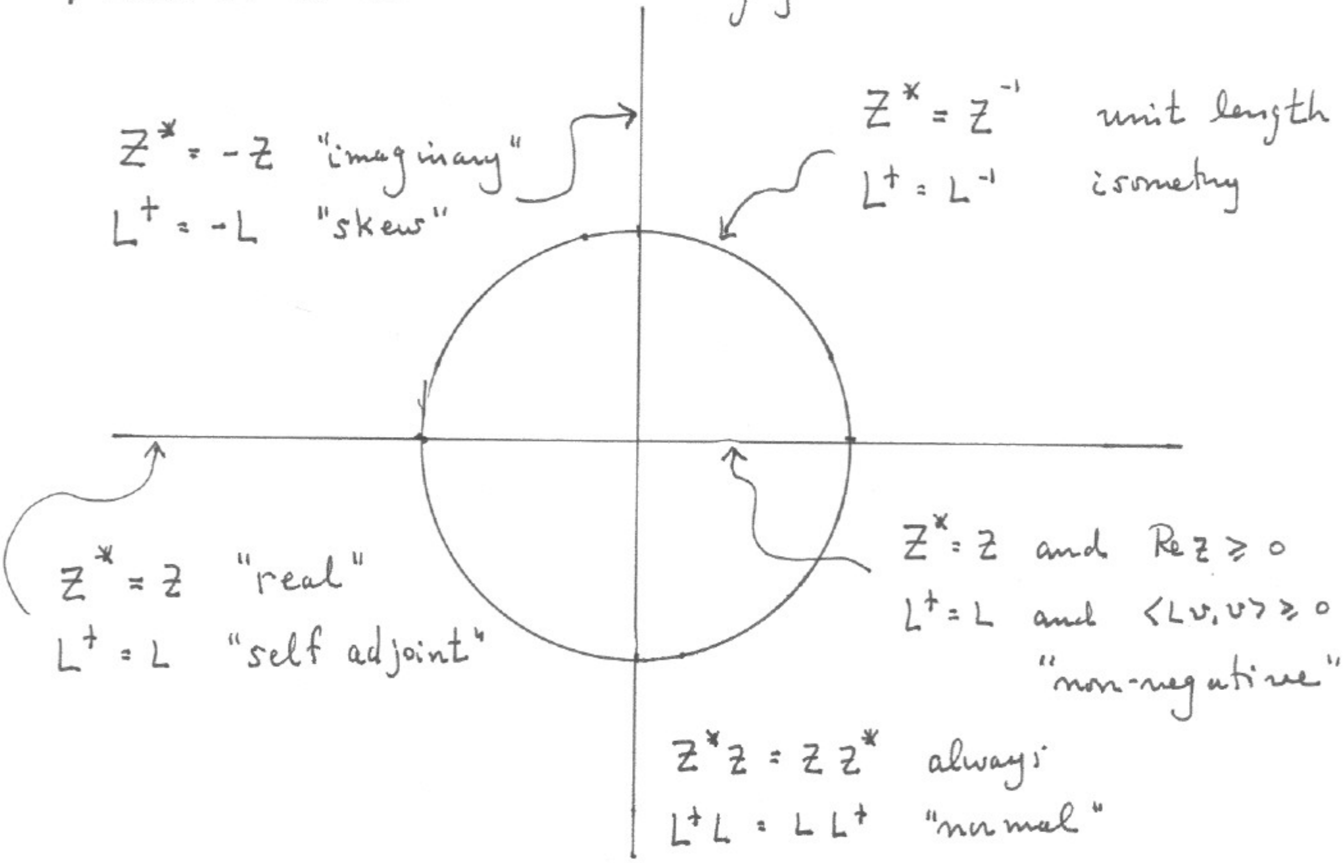
<u>Proof</u>: LL^{+} is self adjoint $\Rightarrow$ has an inverse. $(LL^{+})^{-1} = L^{+^{-1}}L^{-1} = 1$

$\Rightarrow LL^{+} = L^{+}L = 1.$

<u>Fact</u>: L^{+} is an isometry.

<u>Proof</u>: $\|L^{+}v\| = \langle L^{+}v, L^{+}v\rangle = \langle v, LL^{+}v\rangle = \langle v, v\rangle = \|v\|.$

There is a beautiful analogy:



You can use this to guess what's true about operators.

<u>example</u>: If z is positive, z has a positive square roots

<u>Theorem</u>: Every positive operator on V has a unique positive square root.

<u>example</u>: $z = e^{i\theta} (z^* z)^{1/2}$ for some θ.

<u>Theorem</u>: Any operator L can be written

$$L = S \, (L^+ L)^{1/2}$$

where S is an isometry.

See Axler for proofs.