Vector Spaces I Saul Youssef, June 2004

A vector space over a field F is an abelian group V with biadditive $F \times V \to V$ satisfying $a \cdot (b \cdot v) = (ab) \cdot v$ and $1 \cdot v = v$.

ex: $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathrm{Mor}_{set}(A, \mathbb{R})$, n-degree polynomials over $\mathbb{R}$, ...

In practice, F is almost always $\mathbb{R}$ or $\mathbb{C}$. For the following discussion choose a fixed F.

Let a morphism from F-linear space V to F-linear space W be a function $\varphi : V \to W$ which is both a group homomorphism

$$\varphi(v + v') = \varphi(v) + \varphi(v')$$

and a "morphism of scalar multiplication"

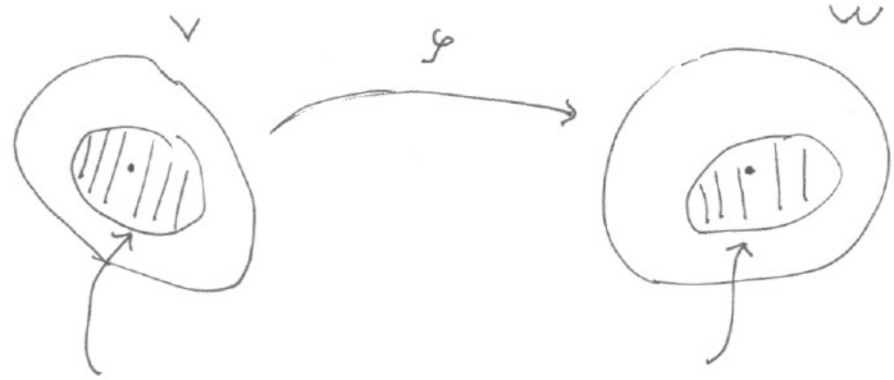
$$\varphi(a \cdot v) = a \cdot \varphi(v)$$

These are called linear mappings. You can easily check that identity mappings and any composition of linear maps is another linear map $\Rightarrow$ we have the category of F-linear spaces for each F.

$$\varphi \in \mathrm{Mor}(V, V)$$

are sometimes called "operators".

Just as with groups, good thing happen with the right choice of morphism.



$\mathrm{Ker}\,\varphi \equiv \{ v \in V : \varphi(v) = 0 \}$ $\qquad$ $\mathrm{Im}\,\varphi \equiv \{ \varphi(v) : v \in V \}$

are both vector spaces, $\varphi(0) = 0$ and $\varphi(-v) = -\varphi(v)$, just as with groups.

You can also easily verify

$$\boxed{\begin{array}{lll} \varphi \text{ is monic} & \Leftrightarrow \varphi \text{ is } 1\text{-}1 & \Leftrightarrow \mathrm{Ker}\,\varphi = \{0\} \\ \varphi \text{ is epic} & \Leftrightarrow \varphi \text{ is onto} & \Leftrightarrow \mathrm{Im}\,\varphi = W \\ \varphi \text{ is iso} & \Leftrightarrow \varphi \text{ is } 1\text{-}1 \text{ and onto} & \Leftrightarrow \mathrm{Ker}\,\varphi = \{0\} \text{ and } \mathrm{Im}\,\varphi = W \end{array}}$$

(see homework).

This is all the same as in group.

Products and Sums

$$\begin{tikzcd} V & V \times W \arrow[l, "\alpha"'] \arrow[r, "\beta"] & W \\ & \mathcal{U} \arrow[ul, "\varphi"] \arrow[u, "\gamma"'] \arrow[ur, "\psi"'] & \end{tikzcd}$$

$V \times W \equiv$ cartesian product with $(v,w) + (v',w') \equiv (v+v', w+w')$

$$a \cdot (v,w) \equiv (av, aw).$$

$V \times W$ is, thus, a vector space and the usual

$$\gamma : u \mapsto (\varphi(u), \psi(u))$$

is a linear mapping for

$$\gamma(u+u') \equiv (\varphi(u+u'), \psi(u+u')) = (\varphi(u) + \varphi(u'), \psi(u) + \psi(u'))$$

$$= (\varphi(u), \psi(u)) + (\varphi(u'), \psi(u')) = \gamma(u) + \gamma(u')$$

and $\gamma(a \cdot u) \equiv (\varphi(a \cdot u), \psi(a \cdot u)) = (a\varphi(u), a\psi(u)) = a \cdot (\varphi(u), \psi(u))$

$$= a \cdot \gamma(u).$$

The same vector space $V \times W \equiv V \oplus W$ is actually the direct sum also

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & V \oplus W \arrow[d, "\gamma"] & W \arrow[l, "\beta"'] \arrow[dl, "\psi"] \\ & \mathcal{U} & \end{tikzcd}$$

$\alpha : v \mapsto (v, 0)$

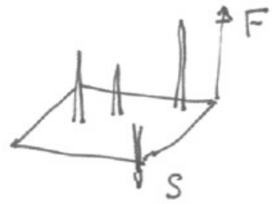
$\beta : w \mapsto (0, w)$

$\gamma : (v,w) \mapsto \varphi(v) + \psi(w)$

Unlike the case of groups, all vector spaces can be created by free construction. Begin with

<u>Theorem:</u> A free vector space exists on any set.

<u>Proof.</u> Let S be a set and let



$$V \equiv \{ F\text{-valued functions } S \to F \text{ which are zero except at a finite number of points} \}$$

V is a vector space in the obvious way with pointwise addition and scalar multiplication: $f, g \in V$, $(f+g)(s) \equiv f(s) + g(s)$, $(a \cdot f)(s) \equiv a \cdot f(s)$.

Let $\alpha(s) \equiv t \mapsto \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise} \end{cases} \in V$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & V \arrow[d, "\bar{\varphi}"] \\ & W \end{tikzcd}$$

The diagram commuting implies $\bar{\varphi}(\alpha(s)) \equiv \varphi(s)$ and since any $v \in V$ is a finite linear combination of $\alpha(s)$, we must have

$$\bar{\varphi}(a_1 \alpha(s_1) + a_2 \alpha(s_2) + \dots + a_n \alpha(s_n))$$

$$= a_1 \varphi(s_1) + a_2 \varphi(s_2) + \dots + a_n \varphi(s_n).$$

Note that this defines a linear function $\Rightarrow$ (α, V) is the free vector space on S.

Independence / Span / Basis

Given a subset S of V, we can form finite "linear combinations" such as

$$a_1 s_1 + a_2 s_2 + \dots + a_n s_n$$

If every element of V is such a combination, we say that S <u>spans</u> V. If

$$a_1 s_1 + a_2 s_2 + \dots + a_n s_n = 0$$

$\Rightarrow a_j = 0$, we say that S is <u>independent</u>. If $S \subset V$ is independent and spans V, we say that S is a <u>basis</u>.

These concepts also nicely characterize monic/epi/iso:

<u>Fact</u>: $V \xrightarrow{\varphi} W$ is monic iff it preserves independence, i.e. if $\varphi[S]$ is independent in W for every independent $S \subset V$.

<u>Fact</u>: $V \xrightarrow{\varphi} W$ is epic iff it preserves span, i.e. if $\varphi[S]$ spans W for every $S \subset V$ which spans V.

<u>Fact</u>: $V \xrightarrow{\varphi} W$ is iso iff φ preserves bases, i.e. if $\varphi[S]$ is a basis for every basis S of V.

<u>Proofs</u>: Easy.

Theorem: Every vector space has a basis.

Proof: Let V be a vector space and consider the collection of independent subsets of V partially ordered by inclusion. If $A_\lambda, \lambda \in \Lambda$ is any totally ordered subset of these, then $\bigcup_\lambda A_\lambda$ is also independent and is an upper bound. By Zorn, there exist maximal independent subsets. Let B be one of these and suppose that B does not span V. However, if v is outside the span of B, $B \cup \{v\}$ is independent $\Rightarrow \Leftarrow$. $\Rightarrow B$ is a basis.

Theorem: Bases of a vector space are isomorphic as sets.

Proof: Geroch theorem 20.

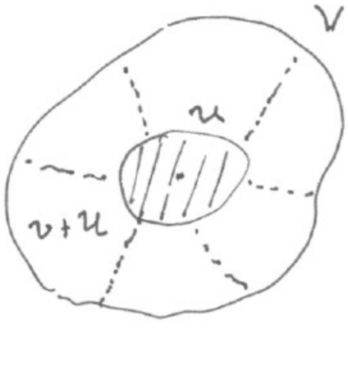
$\Rightarrow$ Every vector space is the free vector space on any of it's sets of basis vectors.

$\Rightarrow$ Every vector space is the free vector space on some set.

$\Rightarrow$ def: The dimension of a vector space is defined to be the size of one of it's bases if the bases are finite. If the bases have infinitely many elements, the space is said to be infinite dimensional.

Subspaces of Vector Spaces

Just as with groups, we define a subspace of a vector space to be a subset which is a vector space on it's own. Just as with groups, we can define...



Subspace $\mathcal{U}$ of V

cosets $v + \mathcal{U}$

- cover V without overlapping
- are isomorphic as sets
- form a vector space "$V/\mathcal{U}$" with

$$(v + \mathcal{U}) + (v' + \mathcal{U}) = (v + v') + \mathcal{U}$$

Also, just as with groups, any morphism $V \xrightarrow{\varphi} W$ can be expanded

$$\begin{tikzcd} V \arrow[r, "\text{epi}"] \arrow[rrr, bend right, "\varphi"'] & V/\operatorname{Ker}\varphi \arrow[r, "\text{iso}"] & \operatorname{Im}\varphi \arrow[r, "\text{mono}"] & W \end{tikzcd}$$

$$\begin{tikzcd} v \arrow[r, mapsto] \arrow[rrr, bend right, mapsto] & v + \operatorname{Ker}\varphi \arrow[r, mapsto] & \varphi(v) \arrow[r, mapsto] & \varphi(v) \end{tikzcd}$$

In vect we say that subspaces $\mathcal{U}, W$ of V are complementary if

(a) Every v can be written $v = u + w$ for some $u \in \mathcal{U}$, $w \in W$.

(b) $u + w = 0 \Rightarrow u = w = 0$.

Fact: If $\mathcal{U}$ and W are complementary in V, then $V \cong \mathcal{U} \oplus W$

Proof: Conditions (a) and (b) above are precisely that the map $\psi : (u, w) \mapsto u + w$ has $\operatorname{Im} \psi = V$, $\operatorname{Ker} \psi = \{0\}$.

Theorem: Every subspace of a vector space has a complementary subspace.

Proof: Note that (b) above is equivalent to $\mathcal{U} \cap W = \{0\}$.

Consider subspaces A of vector space V with the property $\mathcal{U} \cap A = \{0\}$, partially ordered by inclusion. If W_λ is any totally ordered subset of these, $\mathcal{U} \cap (\bigcup_\lambda W_\lambda) = \{0\}$ and $\bigcup_\lambda W_\lambda$ is a subspace of V also. By Zorn, there is a maximal subspace W with $\mathcal{U} \cap W = \{0\}$. Suppose, now, that some $v \in V$ can't be expressed as $v = u + w$, $u \in \mathcal{U}$, $w \in W$. But then $\mathcal{U} \cap [W \cup v] = \{0\}$ because if some element $w + a \cdot v = u \Rightarrow a = 0 \Rightarrow w = u = 0$. This, however, violates the maximality of $W < [W \cup v] \Rightarrow W$ is complementary to $\mathcal{U}$.

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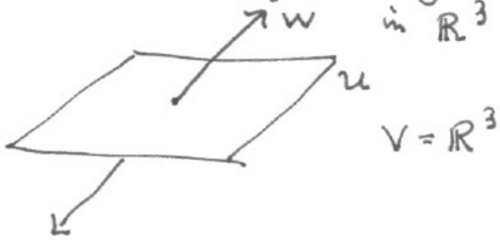
example: $\mathcal{U}$ = a plane through the origin in $\mathbb{R}^3$



any 1-d subspace W not in the plane is complementary

example: $V = \{$ real functions on $[0,1]\}$

$\mathcal{U} \equiv \{$ real functions with $f(x) = 0$ for $x \in [0, 1/2]\}$

$W \equiv \{$ real functions with $f(x) = 0$ on $x \in (1/2, 1]\}$

$\mathcal{U}$ and W are complementary subspaces of V.

Duals

Given a vector space V over F, let

$$V^* \equiv \{ \text{linear maps from } V \text{ to } F \}$$

where, for $f, g \in V$, $(f+g)(v) \equiv f(v) + g(v)$ and $(a \cdot f)(v) = a f(v)$ makes V^* into a vector space also.

<u>example 1</u>: Let $V = \mathbb{R}^3$,

$$dx : (x, y, z) \mapsto x$$
$$dy : (x, y, z) \mapsto y \qquad dx, dy, dz \in V^*$$
$$dz : (x, y, z) \mapsto z$$

In fact, $\{ dx, dy, dz \}$ is a basis of $(\mathbb{R}^3)^*$.

<u>example 2</u>: Let V be the vector space of continuous real valued functions on $[0, 1]$. Fix $p \in V$, then

$$f \mapsto \int_0^1 p(x) f(x)\, dx$$

is an element of V^*.

For every $V \xrightarrow{\varphi} \mathcal{U}$, there is a naturally induced $V^* \xleftarrow{\varphi^*} \mathcal{U}^*$ defined by

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d] & \mathcal{U} \arrow[d, "f"] \arrow[dl, dashed, phantom, "\xleftarrow{\varphi^*}"] \\ F & F \end{tikzcd} \qquad \begin{array}{l} \varphi^* : \mathcal{U}^* \to V^* \\ \varphi^* : f \mapsto f \circ \varphi \end{array}$$

You can also easily verify that "$*$" interacts nicely with composition

$$\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\overset{\varphi^*}{\longleftarrow}"] & \mathcal{U} \arrow[r, "\psi"] \arrow[d] \arrow[dr, phantom, "\overset{\psi^*}{\longleftarrow}"] & W \arrow[d] \\
F & F & F
\end{tikzcd} \qquad (\psi\circ\varphi)^* = \varphi^*\circ\psi^*$$

Note that $\mathbb{R}^3 \cong (\mathbb{R}^3)^*$ in example 1. Is this true in general?

It is almost obvious that $V \cong V^*$ iff V is finite dimensional if you notice that for free $S \to V$ (guaranteed to exist),

$$\begin{tikzcd}
S \arrow[r, "\alpha"] \arrow[dr, "f"'] & V \arrow[d, "\bar{f}"] \\
 & F
\end{tikzcd} \qquad f \longleftrightarrow \bar{f} \text{ is a set isomorphism}$$

so

$$V \cong \{\text{Functions from } S \text{ to } F \text{ which are zero except at a finite number of points}\}$$

$$V^* \cong \{\text{Functions from } S \text{ to } F \text{ in general}\}$$

i.e. ~~t~~ V^* is "bigger" than V. If we knew that V^* can't be isomorphic with a proper subset like V, then we would be done. This, however, is false. (see problems).

<u>example</u>: Let $V \equiv \{$ continuous real functions on $[0,1]\}$

$$\Psi(p) \equiv \left(f \mapsto \int_0^1 p(x)\, f(x) \right) \qquad \Psi : V \to V^*$$

as in example 2. Ψ is 1-1, however, there are elements in V^* which are not in the image of Ψ. For example

$$\delta_y : f \mapsto f(y)$$

is certainly in V^* and can't be expressed as above. These are the "Dirac delta functions".

<u>example</u>: In Dirac's notation, $|x\rangle$ is a state vector and $\langle x|$ is a dual vector. You have to be careful about this because this implies $V \cong V^*$ which is false if V is infinite dimensional.

Multilinear maps and Tensors

In vector space applications one is often dealing with bilinear functions rather than morphisms, e.g.

$$\mathrm{dot}(v, w) \equiv v_x w_x + v_y w_y + v_z w_z \quad \text{in } \mathbb{R}^3$$

$$g(v, w) \equiv v_x w_x + v_y w_y + v_z w_z - v_t w_t \quad \text{in } \mathbb{R}^4$$

area$(v, w) \mapsto$ area in the dotted lines

$$\mathrm{cov}(v, w) \equiv \tfrac{1}{2}\, v^T C w \quad \text{"covariance" in probability.}$$

These are all maps having the property

$$\mu(v + a v', w) = \mu(v, w) + a \cdot \mu(v', w)$$

$$\mu(v, w + b w') = \mu(v, w) + b\, \mu(v, w')$$

$\mu : V \times W \to Z$ $\quad V, W, Z$ vector spaces

cartesian product only, not direct product.

Of course, $V \times W$ is a set and we can always do

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[d, "\bar{\mu}"] \\ & Z \end{tikzcd}$$

where $V \times W \xrightarrow{\alpha} F$ is the free (∞ dimensional) space.

Now we can "simplify F by making use of properties of μ".

Since $\bar{\mu}$ is going to zero all elements of F of the form

$$\alpha(v+av', w) - \alpha(v,w) - a\,\alpha(v',w)$$

$$\alpha(v, w+aw') - \alpha(v,w) - a\,\alpha(v,w')$$

we might as well quotient F with the subspace generated by such elements "A".

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[r, "\beta"] \arrow[d, "\bar{\mu}"] & F/A \arrow[dl, "\bar{\bar{\mu}}"] \\
& Z &
\end{tikzcd}$$

$\beta : f \mapsto f + A$ epic

$A \subset \operatorname{Ker} \bar{\mu}$

$\bar{\bar{\mu}}$ is unique because β is epi (why?)

$F/A \equiv$ "$V \otimes W$" $\qquad \beta \circ \alpha \equiv \otimes$

This is called the tensor product of V and W.

More next time ...

- More tensors
- Exterior algebra, determinants
- Eigenvalue problems, operator algebra
- Inner product spaces
- Applications