Geroch Ch. 3, 4, 5, 6, 7

Saul Youssef
Mathematical Physics 2004

Groups : $(\mathbb{Z},+)$ $(\mathbb{R},+)$, $\mathbb{R}^*$, $\mathbb{C}$, Perm (X), $GL(2,\mathbb{R})$, $SU(2)$, ...

A group is a set with a multiplication $G \times G \to G$ which is...

(a) associative: $g(g'g'') = (gg')g''$

(b) has identities: $ge = eg = g$

(c) has inverses: $gg^{-1} = g^{-1}g = e$

(you should check that inverses and the identity are unique).

Groups where $gg' = g'g$ are called Abelian and the multiplication is traditionally "+" and the identity is traditionally denoted "0".

Groups come up all the time because every "persistent object" in the colloquial sense () gives a group. Also any object A in the categorical sense gives a group

$\mathrm{Aut}(A) \equiv$ group of isomorphisms of A with composition as group multiplication.

ex. In set, Aut(A) is the group of permutations of A.

To make a category, we need the "right" morphisms to go with this structure. The answer is

$$G \xrightarrow{\varphi} H \qquad \varphi(gg') = \varphi(g)\cdot\varphi(g')$$

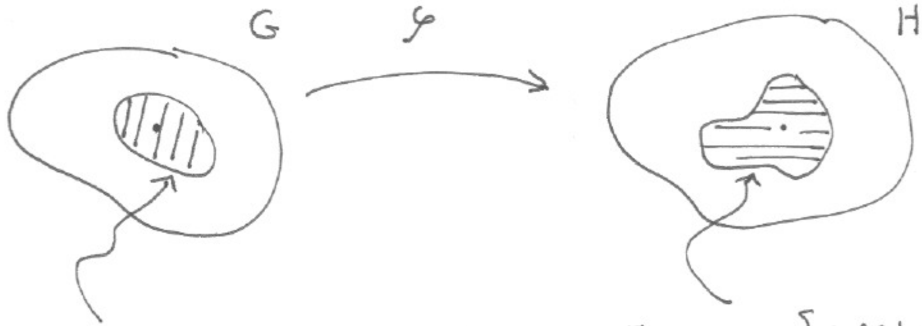
"group homomorphisms." Check that this is a category.

Basic facts about morphisms in groups. $G \xrightarrow{\varphi} H$

$\varphi(e_G) = e_H$ [because $\varphi(e_G)\varphi(e_G) = \varphi(e_G)$]

$\varphi(g^{-1}) = \varphi(g)^{-1}$ [because $\varphi(g)\varphi(g^{-1}) = \varphi(g g^{-1}) = e_H$]

$\varphi : g \mapsto e_H$ is always a group homomorphism.



$\operatorname{Ker}\varphi \equiv \{ g : \varphi(g) = e_H \}$ $\qquad$ $\operatorname{Im}\varphi \equiv \{ \varphi(g) : g \in G \}$

... both are subgroups.

Naturally occurring morphisms;

$L_g : x \mapsto g x$ "left multiplication"

This is actually only a function in general. However it clearly has an inverse and you can check that

$$g \mapsto L_g$$

is a group homomorphism from G to $\operatorname{Perm} G$. This is also clearly monic, so we have proved that

"every group is a subgroup of the permutation group on some set." (Geroch).

Given any group G, we can make a large supply of morphisms by defining "conjugations"

$$C_g : x \mapsto g x g^{-1} \qquad G \to G$$

You can check that this is a morphism (because $(gg')^{-1} = g'^{-1} g^{-1}$).
Notice also that the C_g are closed under composition and have inverses.

$$\{ C_g : g \in G \} \hookrightarrow \mathrm{Aut}\, G$$

are called the group of "inner automorphisms." These are also known as "similarity transformations" in the Physics literature.

From the first lecture, we have several natural categorical questions:

Which morphisms are monic / epi / iso?

Is there a direct product?

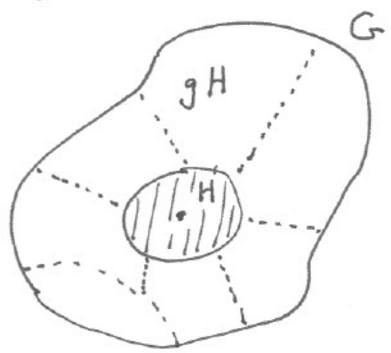
Is there a direct sum?

Plus one more new categorical concept

"Free construction"

First, however, let's analyze subgroups...

Subgroups



How does subgroup H interact with the rest of G?

$$gH \equiv \{ gh : h \in H \}$$

a "left coset" of H.

Theorem: Each $g \in G$ is in exactly one left coset of H.

Proof: Clearly $g \in gH$. If $g \in g'H$ also, then $g = g'h$ for some $h \in H \Rightarrow g'H = g'hH = gH$.

Can we make the left cosets of H into a group?

Try $(gH)\cdot(g'H) \equiv gHg'H$

but this might not be a left coset.

Suppose that H has the property that $Hg' = g'H$ for all $g' \in G$. Then we're ok,

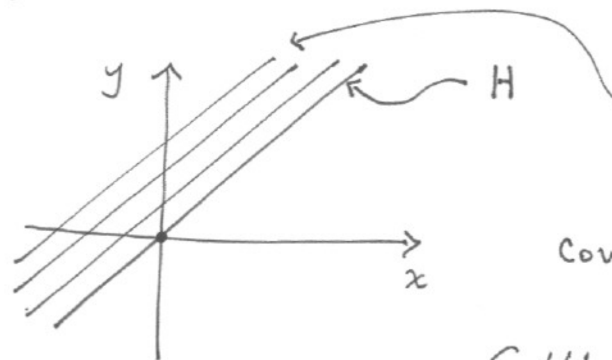
another left coset.

$$(gH)\cdot(g'H) \equiv gHg'H = gg'HH = (gg'H)$$

We say that H is a "normal subgroup" if it has this property. The cosets "G/H" are a group with identity eH and inverse of gH $g^{-1}H$.

[Why not just define $(gH)\cdot(g'H) \equiv gg'H$?]

For example, $G = \mathbb{R}^2$ is a group with addition of vectors in $\mathbb{R}^2$



cosets

$(x,y) + H$

cover the plane without overlapping.

G/H is the group of translations perpendicular to H.

G/H is a <u>quotient</u>, a categorical concept that will occur repeatedly. Consider

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & G/H \arrow[d, "\bar{\varphi}"] \\ & K \end{tikzcd}$$

$\alpha : g \mapsto gH$ is the natural map (epi).

Given any group homomorphism φ, can we "extend" it to $G/H \xrightarrow{\bar{\varphi}} K$?

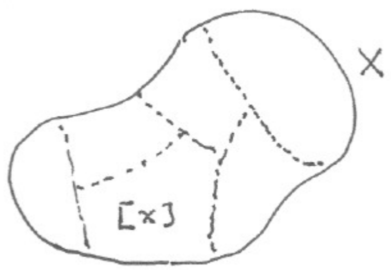
$$\begin{tikzcd} g \arrow[r, mapsto] \arrow[dr, mapsto] & gH \arrow[d, mapsto] \\ & \varphi(g) \end{tikzcd}$$

$\Rightarrow \bar{\varphi}(gH) \equiv \varphi(g)$ is the only possibility. The only issue is whether $\bar{\varphi}$ as defined is a group homomorphism.

For $\bar{\varphi}$ to even be a function, we must have $gH = g'H \Rightarrow \bar{\varphi}(gH) = \bar{\varphi}(g'H)$. This is guaranteed if $\varphi(h) = e$ for any $h \in H$. We also then know that $\bar{\varphi}(gHg'H) = \bar{\varphi}(gg'H) = \varphi(gg') = \bar{\varphi}(gH)\,\bar{\varphi}(g'H)$ is a group homomorphism. ~~It is clearly the unique~~

Notice how very similar group and set are in this respect

Set X with equivalence relation E.



Equivalence classes cover X without overlapping.

X/E is a set.

Any $X \xrightarrow{\varphi} Y$ which "respects the structure"

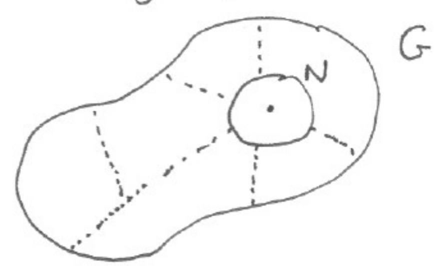
$x E x' \Rightarrow \varphi(x) = \varphi(x')$

$\Rightarrow$

$$\begin{tikzcd} X \arrow[r, "x \mapsto {[x]}"] \arrow[dr, "\varphi"'] & X/E \arrow[d, "\bar{\varphi}"] \\ & Y \end{tikzcd}$$

φ extends to $\bar{\varphi}$ causing the diagram to commute.

Group G with normal subgroup N.



Cosets cover G without overlapping.

G/N is a group.

Any $G \xrightarrow{\varphi} H$ which "respects the structure"

g, g' in the same coset $\Rightarrow \varphi(g) = \varphi(g')$

$\Rightarrow$

$$\begin{tikzcd} G \arrow[r, "g \mapsto gN"] \arrow[dr, "\varphi"'] & G/N \arrow[d, "\bar{\varphi}"] \\ & H \end{tikzcd}$$

φ extends to $\bar{\varphi}$ causing the diagram to commute.

Which morphisms in group are monic/epi/iso?

(a surprisingly hard issue)

<u>Theorem</u>: A group homomorphism $G \xrightarrow{\varphi} H$ is monic iff it is 1-1.

<u>Proof</u>: If φ is 1-1, it is clearly monic in <u>group</u> as well as in <u>set</u>. Conversely, suppose that φ is monic and $\varphi(g) = \varphi(g')$. Then

$$\begin{tikzcd} \{(gg'^{-1})^k : k \in \mathbb{Z}\} \arrow[r, bend left, "g \mapsto g"] \arrow[r, bend right, "g \mapsto e"'] & G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes $\Rightarrow gg'^{-1} = e \Rightarrow g = g' \Rightarrow \varphi$ is 1-1.

<u>Theorem</u>: A group homomorphism $G \xrightarrow{\varphi} H$ is epi iff it is onto.

(Problem #43 in Geroch).

Ker φ and Im φ are nice ways to characterize monic/epi also.

<u>Fact</u>: $G \xrightarrow{\varphi} H$ is monic iff Ker $\varphi = \{e\}$.

<u>Proof</u>: Suppose that Ker φ contains non-identity $g \Rightarrow \varphi(g) = \varphi(e) = e$ and φ is not 1-1 $\Rightarrow$ not monic. Conversely, suppose Ker $\varphi = \{e\}$ and $\varphi(g) = \varphi(g') \Rightarrow \varphi(gg'^{-1}) = e \Rightarrow gg'^{-1} = e \Rightarrow \varphi$ is 1-1 $\Rightarrow \varphi$ is monic.

Clearly φ is epic iff Im $\varphi = H$.

<u>Fact</u>: In the category of groups, φ is iso iff φ is both monic and epic.

<u>Proof</u>. iso $\Rightarrow$ mono and epic in all categories. Suppose that φ is both monic and epic $\Rightarrow$ it has an inverse φ^{-1} as a function. We need to show that φ^{-1} is a morphism in group.

$$\varphi^{-1}(h \cdot h') = \varphi^{-1}(\varphi(g)\varphi(g')) \text{ for some } g, g', \text{ since } \varphi \text{ is onto,}$$

$$= \varphi^{-1}(\varphi(gg')) = gg' = \varphi^{-1}(h) \cdot \varphi^{-1}(h') \Rightarrow \varphi^{-1} \text{ is a group hom.}$$

<u>Summary</u> $G \xrightarrow{\varphi} H$ is

monic	$\Leftrightarrow$ 1-1	$\Leftrightarrow$	Ker $\varphi = \{e\}$
epic	$\Leftrightarrow$ onto	$\Leftrightarrow$	Im $\varphi = H$
iso	$\Leftrightarrow$ both 1-1 and onto	$\Leftrightarrow$	Ker $\varphi = \{e\}$ and Im $\varphi = H$.

Anatomy of a group homomorphism $G \xrightarrow{\varphi} H$

$$\begin{tikzcd} G \arrow[r, "\text{epi}"] \arrow[rrr, bend right, "\varphi"'] & G/\mathrm{Ker}\,\varphi \arrow[r, "\text{iso}"] & \mathrm{Im}\,\varphi \arrow[r, "\text{mono}"] & H \end{tikzcd}$$

$$g \longmapsto g\,\mathrm{Ker}\,\varphi \longmapsto \varphi(g) \longmapsto \varphi(g)$$

You should verify

- That $\mathrm{Ker}\,\varphi$ is a normal subgroup
- That $g \mapsto g\,\mathrm{Ker}\,\varphi$ is an epimorphism
- That $g\,\mathrm{Ker}\,\varphi \mapsto \varphi(g)$ is an isomorphism

Again, there is a very similar construction in <u>set</u>. $X \xrightarrow{\varphi} Y$

let $E_\varphi \equiv \{ (x,x') : \varphi(x) = \varphi(x') \}$

$$X \xrightarrow{\text{epi}} X/E_\varphi \xrightarrow{\text{iso}} \mathrm{Im}\,\varphi \xrightarrow{\text{mono}} Y$$

$$x \longmapsto [x] \longmapsto \varphi(x) \longmapsto \varphi(x)$$

Direct Products and Sums

$$
\begin{tikzcd}
G & G \times H \arrow[l, "\alpha"'] \arrow[r, "\beta"] & H \\
 & X \arrow[ul, "\varphi"] \arrow[u, "\gamma"'] \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

guess $G \times H$ = cartesian product with

$(g, h) \cdot (g', h') \equiv (gg', hh')$

$\Rightarrow$ a group.

From set, we already know that the diagram commutes for the unique function $\gamma : x \mapsto (\varphi(x), \psi(x))$. The only issue is whether γ is a group homomorphism.

$\cancel{\gamma(xy}$

$$\gamma(xy) = (\varphi(xy), \psi(xy)) = (\varphi(x)\varphi(y), \psi(x)\psi(y))$$

$$= (\varphi(x), \psi(x))(\varphi(y), \psi(y)) = \gamma(x)\gamma(y) \quad .$$

Direct Sum is harder: done as one of the homework problems.

Free groups is a categorical concept for crossing category boundaries in a systematic way.

Set | Group

$$
\begin{tikzcd}
S \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & G \arrow[d, "\gamma"] \\
 & H
\end{tikzcd}
$$

Let G be simply lists of elements from S with primes to indicate inverses.

$S = \{a, b, c, \dots\}$ $\qquad$ $acb'bcaa' \in G$

where $aa' = a'a =$ ""
$bb' = b'b =$ ""
etc.

This is a group with concatenation as multiplication and the empty list as the identity.

e.g. $(acb'bcaa')^{-1} = aa'c'b'bc'a'$.

This will seem less arbitrary if you realize that...

Theorem: Free groups are unique.

Proof: Similar to direct product uniqueness proof.

Proof. Suppose that both $S \xrightarrow{\alpha} V$ and $S \xrightarrow{\alpha'} V'$ are free $\mathfrak{k}$ groups on S.

$$\begin{tikzcd}
S \arrow[r, "\alpha"] \arrow[d, "i_S"'] & V \arrow[d, "\gamma"] \\
S \arrow[r, "\alpha'"] \arrow[d, "i_S"'] & V' \arrow[d, "\gamma'"] \\
S \arrow[r, "\alpha"] & V
\end{tikzcd}$$

commutes

commutes

$$\Rightarrow \quad \begin{tikzcd}
S \arrow[r, "\alpha"] \arrow[d, "i_S"'] & V \arrow[d, "\gamma'\circ\gamma"] \\
S \arrow[r, "\alpha"] & V
\end{tikzcd} \quad \text{commutes} \Rightarrow \gamma'\circ\gamma = i_V$$

$$\Rightarrow \gamma\circ\gamma' = i_{V'} \quad \Rightarrow \quad V \cong V'.$$

Applications

Many familiar functions are group morphisms, e.g.

$$\exp : \mathbb{R} \to \mathbb{R}^{>0} \qquad \log : \mathbb{R}^{>0} \to \mathbb{R}$$

$$\mathrm{abs} : \mathbb{R}^{\neq 0} \to \mathbb{R}^{>0} \qquad \mathrm{sign} : \mathbb{R}^{\neq 0} \to \{-1, +1\}$$

$$\mathrm{fraction} : \mathbb{R} \to (-1, +1) \qquad \mathrm{mod}_2 : \mathbb{Z} \to \mathbb{Z}_2$$

$$\det : GL(n) \to \mathbb{R}^{\neq 0} \qquad \mathrm{Re} : \mathbb{C} \to \mathbb{R}$$

Just recognizing a morphism can imply alot

$$\begin{tikzcd} \mathcal{O}(3) \arrow[r, "\det"] \arrow[rr, bend right, "\equiv \varphi"'] & \mathbb{R}^{\neq 0} \arrow[r, "\mathrm{sign}"] & \{-1, +1\} \end{tikzcd}$$

Implies....

$\mathrm{Ker}\,\varphi \equiv SO(3)$ is a normal subgroup of $\mathcal{O}(3)$.

$SO(3)$ has exactly one coset $\equiv SO(3)^-$ "Inversions"

$SO(3) \cong SO(3)^-$ as a set.

If $g \in SO(3)$, $x g x^{-1} \in SO(3)$ for any $x \in \mathcal{O}(3)$.

$SO(3) \cdot SO(3) = SO(3)$

$SO(3) \cdot SO(3)^- = SO(3)^-$

$SO(3)^- \cdot SO(3) = SO(3)^-$

$SO(3)^- \cdot SO(3)^- = SO(3)$

In general $G \longrightarrow \{-1, +1\}$ gives even/odd structure.

... Just from the existence of morphism $\varphi = \mathrm{sign} \circ \det$.

The decomposition of $G \xrightarrow{\varphi} H$

$$\begin{tikzcd} G \arrow[r, "\text{epi}"] \arrow[rrr, bend right, "\varphi"'] & G/\mathrm{Ker}\,\varphi \arrow[r, "\text{iso}"] & \mathrm{Im}\,\varphi \arrow[r, "\text{mono}"] & H \end{tikzcd}$$

gives great tricks for answering questions like what is $\mathbb{R}^{\neq 0}/\{-1,+1\}$

- Try to guess a morphism $\mathbb{R}^{\neq 0} \xrightarrow{\varphi} ?$ so that $\mathrm{Ker}\,\varphi = \{-1,+1\}$.

$$\begin{tikzcd} \mathbb{R}^{\neq 0} \arrow[r] \arrow[dr, "\mathrm{abs}"'] & \mathbb{R}^{\neq 0}/\mathrm{Ker}(\mathrm{abs}) \arrow[d, "\overline{\mathrm{abs}}"] \\ & \mathbb{R}^{>0} \end{tikzcd} \qquad \mathrm{Ker}(\mathrm{abs}) = \{-1,+1\}$$

By the decomposition, $\overline{\mathrm{abs}}$ is monic. Since abs is also epi, $\overline{\mathrm{abs}}$ is epi as well $\Rightarrow$ iso $\Rightarrow$ $\mathbb{R}^{\neq 0}/\{-1,+1\} \cong \mathbb{R}^{>0}$.

Further reading

Group Theory and Physics, Shlomo Sternberg.*

Elements of Group Theory for Physicists, A.W. Joshi.

Representation Theory: A First Course, Fulton and Harris.

* A really wonderful book.