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Lecture 1.

A category is a collection of "objects" such that for each pair A, B of objects we have a set of morphisms Mor(A,B). Morphisms can be composed

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi \circ \varphi"'] & B \arrow[r, "\psi"] & C \end{tikzcd}$$

The composition must be associative. Also, for each object A, there must be an identity morphism $A \xrightarrow{i_A} A$ such that $\varphi \circ i_A = \varphi$ and $\cancel{\psi \circ i}$ $i_A \circ \psi = \psi$ for any φ, ψ.

examples

objects	morphisms
sets	functions
ordered sets	monotonic functions
groups	"group homomorphisms"
complex vector spaces	complex linear maps
topological spaces	continuous maps
manifolds	smooth maps
pointed sets	$(X,x) \xrightarrow{\varphi} (Y,y)$ s.t. $\varphi(x) = y$
"Principle G-bundles"	$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$
Categories	?
?	"Natural transformations"

ex.
Power set of a set
dual of a vector space
Lie algebra of a Lie Group
Homotopy ⊙ ≠ ◎
cohomology
"second quantization"
...are all the same categorical construction.

Distinguished Morphisms

Category of sets

A morphism $A \xrightarrow{\varphi} B$ is...

... a monomorphism if

$$\begin{tikzcd} X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\alpha'"'] & A \arrow[r, "\varphi"] & B \end{tikzcd} \Rightarrow \alpha = \alpha' \iff$$

φ is one-to-one
φ is "injective"

... an epimorphism if

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\alpha'"'] & X \end{tikzcd} \Rightarrow \alpha = \alpha' \iff$$

φ is onto
φ is "surjective"

... an isomorphism if

$$\begin{tikzcd} i_A \circlearrowright A \arrow[r, bend left, "\varphi"] & B \circlearrowleft i_B \arrow[l, bend left, "\varphi^{-1}"] \end{tikzcd} \iff$$

commutes for some φ^{-1}. "$A \cong B$"

φ has a two sided inverse
φ is "bijective"

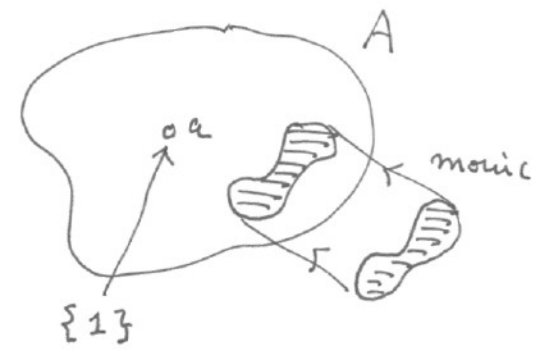
Theorem. A function in set $A \xrightarrow{\varphi} B$ is a monomorphism iff it is 1-1.

Proof. Suppose that φ is monic.

Then

$$\begin{tikzcd} \{1\} \arrow[r, shift left, "a"] \arrow[r, shift right, "a'"'] & A \arrow[r, "\varphi"] & B \end{tikzcd} \Rightarrow a = a'$$

iff $\varphi(a) = \varphi(a') \Rightarrow a = a' \Rightarrow \varphi$ is 1-1.



morphisms as "probes"

Conversely, suppose that φ is 1-1 and

$$\begin{tikzcd} X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\alpha'"'] & A \arrow[r, "\varphi"] & B \end{tikzcd}$$ commutes for some $\alpha \neq \alpha'$.

$\Rightarrow \alpha(x) \neq \alpha'(x)$ for some $x \in X \Rightarrow \varphi(\alpha(x)) \neq \varphi(\alpha'(x))$ since φ is 1-1

$\Rightarrow$ the diagram does not commute $\Rightarrow \Leftarrow$.

Simple facts

Fact. If $A \xrightarrow{\varphi} B$ and $B \xrightarrow{\psi} C$ are monic, so is $\psi \circ \varphi$.

Proof: Consider

$$\begin{tikzcd} X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\alpha'"'] & A \arrow[r, "\varphi"] & B \arrow[r, "\psi"] & C. \end{tikzcd}$$

If this commutes, then $\varphi \circ \alpha = \varphi \circ \alpha'$ (since ψ is monic)
$\Rightarrow \alpha = \alpha'$ (since φ is also monic) $\Rightarrow \psi \circ \varphi$ is monic.

+ miscellaneous similar results.

Products and Sums

Can we "treat a bunch of objects effectively as one object"?

$$\begin{tikzcd} A & B & C & D \\ & X \arrow[ul, "\varphi_1"] \arrow[u, "\varphi_2"] \arrow[ur, "\varphi_3"] \arrow[urr, "\varphi_4"'] \end{tikzcd} \quad \overset{?}{\longleftrightarrow} \quad \begin{tikzcd} A & B & C & D \\ & \mathrm{Prod} \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] \\ & X \arrow[u, "\gamma"] \end{tikzcd} \Big\} \text{ fixed}$$

$(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \longleftrightarrow \gamma$?

A product "$A \times B$" of objects A, B consists of $A \times B$ and two morphisms α, β

$$\begin{tikzcd} A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\ & X \arrow[u, "\gamma"] \arrow[ul, bend left, "\varphi"] \arrow[ur, bend right, "\psi"'] \end{tikzcd}$$

For any X, φ, ψ, there is a unique γ s.t. the diagram commutes.

In the category of sets, $A \times B$ is the cartesian product

$$\begin{tikzcd} \varphi(x) & (\varphi(x), \psi(x)) \arrow[l, mapsto] \arrow[r, mapsto] & \psi(x) \\ & x \arrow[u, mapsto, "\gamma"] \arrow[ul, mapsto, bend left, "\varphi"] \arrow[ur, mapsto, bend right] \end{tikzcd}$$

$\alpha(a,b) \equiv a \quad \beta(a,b) \equiv b$

the diagram commutes for exactly one function

$\gamma : x \mapsto (\varphi(x), \psi(x))$.

<u>Theorem</u>: Products are unique in any category

<u>Proof.</u> Suppose $(A \times B, \alpha, \beta)$ and $((A \times B)', \alpha', \beta')$ are both products of A and B.

$$\begin{tikzcd}
A \arrow[d, "i_A"'] & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] \arrow[d, "\gamma'"] & B \arrow[d, "i_B"] \\
A \arrow[d, "i_A"'] & (A \times B)' \arrow[l, "\alpha'"'] \arrow[r, "\beta'"] \arrow[d, "\gamma"] & B \arrow[d, "i_B"] \\
A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B
\end{tikzcd}$$

commutes for a unique γ'

commutes for a unique γ

$\Rightarrow$ the whole diagram commutes

$\Rightarrow$

$$\begin{tikzcd}
A \arrow[d, "i_A"'] & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] \arrow[d, "\gamma \circ \gamma'"] & B \arrow[d] \\
A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B
\end{tikzcd}$$

commutes

However this clearly also commutes with $i_{A \times B}$.

Since the product $A \times B$ guarantees a <u>unique</u> morphism, $i_{A \times B} = \gamma \circ \gamma'$. Similarly $i_{(A \times B)'} = \gamma' \circ \gamma \Rightarrow A \times B \cong (A \times B)'$.

Misc. definitions in <u>set</u>

A <u>function</u> $A \xrightarrow{f} B$ is often specified informally by a rule such as $f: a \mapsto a^2$. ~~#~~ Strictly speaking a function is a subset of $A \times B$

$$f \equiv \{ (a, f(a)) : a \in A \}$$

A <u>relation</u> R on a set A is just a subset of $A \times A$.

$$\text{"} a R b \text{"} \equiv (a,b) \in R$$

e.x. The partial ordering "$\leq$" is a relation with

$a \leq b$ and $b \leq c \Rightarrow a \leq c$ "transitive"

$a \leq b$ and $b \leq a \Rightarrow a = b$ "symmetric"

$a \leq a$ "reflexive"

For ~~most~~ example, subsets of a set X are "partially ordered by inclusion" if $A \leq B$ iff $A \subset B$, $A, B \subset X$.

An <u>equivalence relation</u> E has these properties

$a E b$ and $b E c \Rightarrow a E c$

$a E b \Rightarrow b E a$

$a E a$

ex. "$=$" is an equivalence relation.

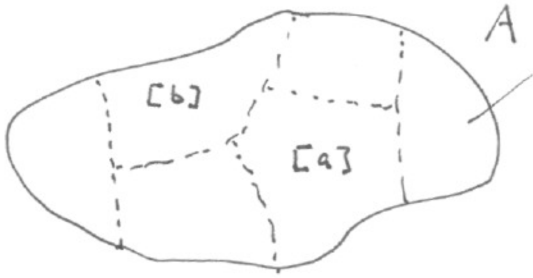
ex. $r E r' \equiv r - r' = n \cdot 2\pi$ is an equivalence relation on $\mathbb{R}$.

The <u>equivalence class</u> containing $a \in A$ is defined to be

$$[a] \equiv \{ x \in A : x E a \}$$

<u>Theorem</u>. Each $a \in A$ is in exactly one equivalence class.

<u>Proof.</u> Let $a \in A$. Then $a \in [a]$ because $a E a$. If $a \in [b]$ also, then $a E b$. $\Rightarrow$ for any $x \in [a]$, $x E a, a E b \Rightarrow x E b \Rightarrow x \in [b]$. Similarly $[b] \subset [a]$. $\Rightarrow [a] = [b]$.



The set of equivalence classes "A/E" cover A without overlapping.

<u>Theorem</u>. If E_λ are equivalence relations on A, then $\bigcap_\lambda E_\lambda$ is also an equivalence relation on A.

<u>Proof.</u> Easy.

<u>Def.</u> The equivalence relation <u>generated</u> by relation $R \subset A \times A$

$$Eq(R) \equiv \{ \text{intersection of all equivalence relations that contain } R \}.$$

<u>example:</u>

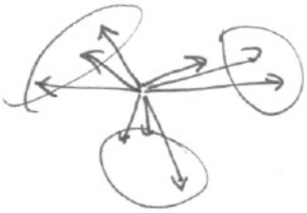
A relation $R \subset A \times A$ is just the (solid) edges of some graph on A. $Eq(R)$ is the relation

$a \, Eq(R) \, b$ iff there is a path from a to b.

$A/Eq(R)$ are just the connected components.

Even in set, there are non-trivial applications

Jet finding in HEP

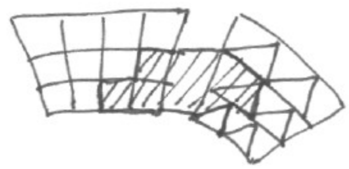


A = Momentum vectors

pRq if q is the "nearest" top

$A/Eq(R)$ ≡ jets

Clustering in a pixel detector

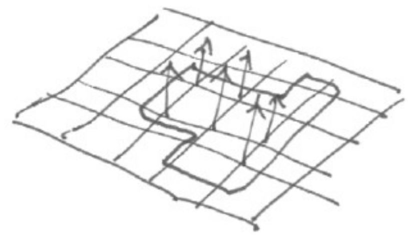


A = Cells

cRd if cell c is the "nearest" to d.

$A/Eq(R)$ ≡ clusters

Finding contiguous spins on a lattice



A = lattice sites

iRj if i and j are adjacent and spin(i) = spin(j).

$A/Eq(R)$ ≡ contiguous spin groups

all are solved by the same $n \log n$ ($n \equiv |A|$) algorithm: "finding the connected components of a digraph".