Homework #1: Groups
Mathematical Physics
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1. Read Geroch's mini-chapters 3, 4, 5, 6 and 7.

2. Recall the definition of a group and prove basic facts: a) identities and inverses are unique, b) $(gh)^{-1} = h^{-1}g^{-1}$, c) if $G \xrightarrow{\varphi} H$ is a homomorphism, $\varphi(e_G) = e_H$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$.

3. (Geroch's Exercise 13): Are the reals with real multiplication a group? Show that the set of all positive reals, with product multiplication, is a group and is isomorphic to the additive group of reals.

4. (Geroch's Exercise 24): Find, for each positive integer n, a group having exactly n elements.

5. (Geroch's Exercise 29): Is the union of subgroups of a group a subgroup?

6. Which groups G are isomorphic to the permutation group on the set G?

7. (Geroch's Exercise 26): Let G be a group, and consider the collection of all subsets of G with product (for A and A' subsets) AA'. Is this a group?

8. Find the free abelian group on a set.

9. (Geroch's Exercise 36): Regarding the additive group of integers as a subgroup of the additive group of reals, find the quotient group.

10. (Geroch's Exercise 37): Show that any subgroup having just two left cosets is normal.

11. Given a group G, show that conjugations (a.k.a. "inner automorphisms" or "similarity transformations") $C_g(x) = g^{-1}xg$ are a normal subgroup of $\mathrm{Aut}(G)$.

12. (Geroch's Exercise 42): Find a nontrivial normal subgroup of the free group on a set with three elements.

13. (Geroch's Exercise 46): Prove that $G \xrightarrow{\varphi} H$ is one-to-one if and only if $\mathrm{Ker}(\varphi)$ contains only e, and onto if and only if $\mathrm{Im}(\varphi) = H$.

14. Let $\mathbf{R}^*$ be the multiplicative group of non-zero reals. What is $\mathbf{R}^*/\{-1, +1\}$?

15. Find the direct sum in the category of groups.