Vector Spaces (S. Youssef, Feb. 2000) Geroch Ch. 9, 10, 11, 13

A real (complex) vector space V is an abelian group with bilinear product $\mathbb{R}\times V \to V$ ($\mathbb{C}\times V\to V$) satisfying $(aa')v = a(a'v)$ and $1\cdot v = v$.

The category of real (complex) vector spaces consists of

Objects: real (complex) vector spaces

Morphisms: <u>linear</u> maps $V \xrightarrow{\varphi} W$ $\quad \varphi(v+av') = \varphi(v) + a^*\varphi(v')$

In this category

monomorphisms $\Leftrightarrow$ one-to-one linear maps $\Leftrightarrow$ $\mathrm{Ker}\,\varphi = \{0\}$

epimorphisms $\Leftrightarrow$ onto linear maps $\Leftrightarrow$ $\mathrm{Im}\,\varphi = W$

isomorphisms $\Leftrightarrow$ bi-invertible linear maps $\Leftrightarrow$ $\mathrm{Ker}\,\varphi = \{0\}$ and $\mathrm{Im}\,\varphi = W$.

The cartesian product $V\times W$ with the obvious vector space structure is denoted $V\oplus W$ and is both direct sum and direct product

$\mathrm{Mor}(V,W)$ is also a vector space with $(f+g)(v) = f(v) + g(v)$ etc.

A free construction

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & V \\ & \searrow^{\varphi} & \downarrow \gamma \\ & & W \end{array}$$

$\alpha : s \mapsto \left(s' \mapsto \begin{matrix} 1 \text{ if } s'=s \\ 0 \text{ if } s'\neq s\end{matrix}\right)$

$\gamma(f) = \overrightarrow{\sum_{s\in S}} \varphi(s) f(s)$

V: vector space of functions from S to $\mathbb{R}$ zero except at a finite number of points

Subset K of V is <u>independent</u> if $a_j k_j = 0 \Rightarrow a_j = 0$ [finite sum over repeated indices]

Subset K of V <u>spans V</u> if $v = a_j k_j$ for all $v\in V$.

Subset K of V is a <u>basis</u> if K is independent and spans V.

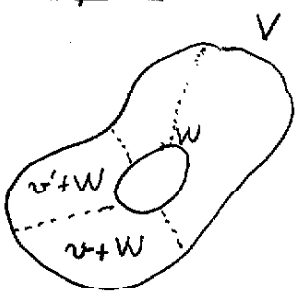
A vector space V has a basis $\Leftrightarrow$ V is the free vector space on some set.

Theorem: Every vector space has a basis.

Theorem: For free vector spaces $S\to V$ and $S'\to V'$, $V\cong V' \Rightarrow S\cong S'$.

} uses Zorn's lemma

Subspaces



Just as with groups, $W \subset V$ is called a <u>subspace</u> if it is a vector space on its own. Also, just as with groups, for $V \xrightarrow{\varphi} \mathcal{U}$, $\operatorname{Ker}\varphi$ and $\operatorname{Im}\varphi$ are subspaces. All subspaces are normal, so V/W always exists.

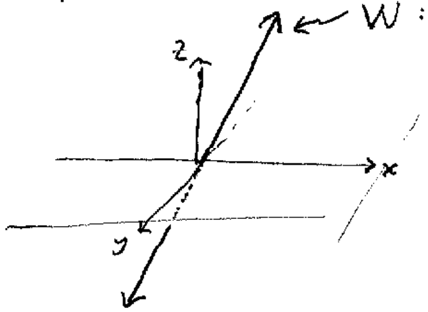
$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & V/W \arrow[d, "\gamma"] \\ & Z \end{tikzcd}$$

If $W \subset \operatorname{Ker}\varphi$, there is a unique γ s.t. the diagram commutes.

If $W = \operatorname{Ker}\varphi$, then γ is also a monomorphism. If φ is also epi, then $V/W \cong Z$

example: $V = \mathbb{R}^3$

W : a subspace



V/W is the set of lines parallel to W

$V/W \cong$ x-y plane

Subspaces $\mathcal{U}$ and W are <u>complementary</u> in V if, for every $v \in V$, $v = u + w$ for some $u \in \mathcal{U}, w \in W$, <u>and</u> if $u + w = 0 \Rightarrow u = w = 0$.

Theorem (Geroch): Complementary subspaces exist.

Theorem: If $\mathcal{U}$ and W are complementary in V, then $V \cong \mathcal{U} \oplus W$.

Proof. Let linear $\varphi : (u, w) \mapsto u + w$ be a mapping from $\mathcal{U} \oplus W$ to V. Since $\mathcal{U}$ and W are complementary, φ is epi. Suppose $\varphi((u,w)) = \varphi((u',w'))$. Then $u + w = u' + w' \Rightarrow (u - u') + (w - w') = 0$ $\Rightarrow u = u', w = w' \Rightarrow \varphi$ is mono $\Rightarrow V \cong \mathcal{U} \oplus W$.

example: $V = \operatorname{Mw}([0,1], \mathbb{R})$, $A \subset [0,1]$

$\mathcal{U} = \{ f \in V : f(x) = 0 \text{ for } x \in A \}$

$W = \{ f \in V : f(x) = 0 \text{ for } x \notin A \}$

$\mathcal{U}$ and W are complementary

Geroch # 83.

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & V \arrow[d, dashed, "??"] \\ & W \end{tikzcd}$$

Given α, β, is there a morphism as indicated that causes the triangle to commute?

If so, when is it unique?

<u>Answer</u>: Such a morphism exists if and only if $\operatorname{Ker}\alpha \subset \operatorname{Ker}\beta$.

<u>Proof</u>. Suppose $V \xrightarrow{\gamma} W$ causes the diagram to commute. Then $\operatorname{Ker}\beta = \operatorname{Ker}(\gamma\circ\alpha) \supset \operatorname{Ker}\alpha$.

Suppose that $\operatorname{Ker}\alpha \subset \operatorname{Ker}\beta$. Define a morphism $\operatorname{Im}\alpha \xrightarrow{\lambda} \operatorname{Im}\beta$ by

$$\lambda : \alpha(u) \mapsto \beta(u)$$

Notice that $\alpha(u) = \alpha(u') \Rightarrow u - u' \in \operatorname{Ker}\alpha \Rightarrow u - u' \in \operatorname{Ker}\beta$
$\Rightarrow \beta(u) = \beta(u') \Rightarrow \lambda(\alpha(u)) = \lambda(\alpha(u')) \Rightarrow \lambda$ is a function.
λ is also linear since $\lambda(\alpha(u) + a(\alpha(u'))) = \lambda(\alpha(u + au'))$
$= \beta(u + au') = \beta(u) + a\beta(u') = \lambda(\alpha(u)) + a\lambda(\alpha(u'))$.

Let $V \cong \operatorname{Im}\alpha \oplus V'$ where V' is complementary to $\operatorname{Im}\alpha$
$W \cong \operatorname{Im}\beta \oplus W'$ where W' is complementary to $\operatorname{Im}\beta$

Consider

$$\begin{tikzcd} \operatorname{Im}\alpha \arrow[r] \arrow[d, "\lambda\cancel{\gamma}"'] & \operatorname{Im}\alpha \oplus V' \arrow[d, "\gamma"] & V' \arrow[l] \arrow[d, "x"] \\ \operatorname{Im}\beta \arrow[r] & \operatorname{Im}\beta \oplus W' & W' \arrow[l] \end{tikzcd}$$

Since there is always a morphism $x : v' \mapsto 0_{W'}$, γ solves the problem. Also, there will be more than one γ unless there is only one $x : V' \to W'$, e.g. if $V' = \{0\}$ or $W' \{0\}$ e.g. unless α or β are epi.

If β is epi, γ is unique. Also, if α is epi, λ is uniquely determined by the triangle commuting. QED

Duals

Given a vector space V, $\mathrm{Mor}(V, \mathbb{R})$ ("V^*") is called the dual of V. Notice that because of the free construction,

$$\begin{tikzcd} S \arrow[r, "d"] \arrow[dr, "f"'] & V \arrow[d, "\varphi"] \\ & \mathbb{R} \end{tikzcd}$$

$f \mapsto \varphi$ is an isomorphism $\mathrm{Mor}(S, \mathbb{R}) \cong V^*$.

Given $V \xrightarrow{\varphi} W$, the dual lets us also define $V^* \xleftarrow{\varphi^*} W^*$ by

$$\varphi^* : f \mapsto f \circ \varphi$$

example: Let V be the vector space of continuous functions from $[0,1]$ to $\mathbb{R}$. For each $m \in V$, we can define

$$f \mapsto \int_0^1 f(x)\, m(x)\, dx \qquad V \to \mathbb{R} \quad \text{i.e.} \in V^*$$

V is isomorphic to such elements, but this is not all of V^*!

For example for any fixed $a \in [0,1]$,

$$f \mapsto f(a)$$

is linear, but this element of V^* is not $\int_0^1 f(x)\, m(x)\, dx$ for any m ~~[illegible]~~ [it works for $m(x) =$ "$\delta(x-a)$", but $\delta(x-a)$ is not a function].

For finite dimensional vector spaces, $V \cong V^*$. In general, Geroch proves that $V \cong V^{**}$ iff V is finite dimensional.

Geroch #89.

Show that $(V \oplus W)^* \cong V^* \oplus W^*$.

Consider the direct sum

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, bend right, "f"'] & V \oplus W \arrow[d, "\gamma"] & W \arrow[l, "\beta"'] \arrow[dl, bend left, "g"] \\ & \mathbb{R} & \end{tikzcd}$$

This constitutes an invertible map $\Psi : (f, g) \mapsto \gamma$ from $V^* \oplus W^*$ to $(V \oplus W)^*$. Ψ is also linear, so $V^* \oplus W^* \cong (V \oplus W)^*$.

example.

Suppose that $V \xrightarrow{\varphi} W$ is a monomorphism.
We want to show that $\varphi^*: W^* \to V^*$, $\varphi^*: f \mapsto f \circ \varphi$
is epi. Let g be any element of V^*.

Since φ is mono, $\operatorname{Ker} \varphi \subset \operatorname{Ker} g$, a $\bar{g}$ exists s.t.

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "g"'] & W \arrow[d, "\bar{g}"] \\ & \mathbb{R} \end{tikzcd}$$

commutes. Then $g = \varphi^*(\bar{g}) \Rightarrow \varphi^*$ is epi.

On the other hand, if φ is epi, then ~~[illegible]~~ suppose $\varphi^*(\alpha) = \varphi^*(\alpha')$. Then

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[dr, "\varphi^*(\alpha)"'] & W \arrow[d, bend right, "\alpha"'] \arrow[d, bend left, "\alpha'"] \\ & \mathbb{R} \end{tikzcd}$$

commutes. $\Rightarrow \alpha = \alpha' \Rightarrow \varphi^*$ is monomorphic.

Tensor Products (Defined in Geroch).

Given multilinear $V \times W \xrightarrow{\mu} Z$, let $V \times W \xrightarrow{\alpha} F$ be the free vector space on $V \times W$ and let A be the subspace generated by

$$\left.\begin{aligned} &\alpha(v+av', w) - \alpha(v, w) - a\,\alpha(v', w) \\ &\alpha(v, w+aw') - \alpha(v, w) - a\,\alpha(v, w') \end{aligned}\right\} \bar{A}$$

for $v, v' \in V$, $w, w' \in W$, $a \in \mathbb{R}$. Consider

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[d, "\gamma"] \\ & Z \end{tikzcd}$$

Since γ is linear and $\gamma(a) = 0$ for any $a \in \bar{A}$, γ is also zero on any $a' \in A$.

Thus

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[r, "\beta"] \arrow[d, "\gamma"] & F/A \arrow[dl, "\gamma'"] \\ & Z & \end{tikzcd}$$

$A \subset \operatorname{Ker} \gamma \Rightarrow$ there is a γ' s.t. the right triangle commutes. Since β is epi, γ' is unique. $\Rightarrow$ The whole diagram commutes.

It only remains to show that $\beta \circ \alpha$ is bilinear

$$\beta\circ\alpha(v+av', w) - \beta\circ\alpha(v, w) - a\,\beta\circ\alpha(v, w')$$

$$= \beta\Big(\underbrace{\alpha(v+av', w) - \alpha(v, w) - a\,\alpha(v, w')}_{\in A}\Big) = 0.$$

Thus $(\beta\circ\alpha, F/A)$ is the tensor product of $V \times W$. This is conventionally renamed

$$V \times W \xrightarrow{\otimes} V \otimes W.$$