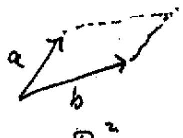
S. Y., Feb 2000

In vector space applications one often deals with functions which are not morphisms. For example...

$$\mathrm{dot}(v,v') \equiv v_x v'_x + v_y v'_y + v_z v'_z \qquad \text{euclidean metric}$$

$$g(v,v') \equiv v_x v'_x + v_y v'_y + v_z v'_z - v_t v'_t \qquad \text{lorentzian metric}$$



area : $(a,b) \mapsto$ area in the dotted lines

$\frac{1}{2} x^t M x$ "covariance matrix" in probability

These have in common that they are "bi-linear" mappings.
In general, a bilinear map $\mu : V \times W \to Z$ is linear in both arguments separately

$$\mu(v + av', w) = \mu(v,w) + a\mu(v',w) \qquad \text{for all } a \in \mathbb{R}$$

$$\mu(v, w + aw') = \mu(v,w) + a\mu(v,w') \qquad v, v' \in V \quad w, w' \in W$$

There is an obvious generalization of this to "multilinear" functions.

Of course, $V \times W$ is a set and we can extend μ into a morphism using the free construction

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[d, "\bar{\mu}"] \\ & Z \end{tikzcd}$$

F is the free vector space on $V \times W$.

However, we have not yet made use of μ's special properties and, as a result, F is bigger than it has to be.
[F is actually infinite dimensional, but this is no problem for us]

As usual, to simplify F and $\bar{\mu}$, we want to find a subspace of F, take the quotient and use the standard quotient triangle diagram to extend $\bar{\mu}$ to the quotient space. For the quotient construction to work, we want $\bar{\mu}$ to zero the subspace. To make $F/?$ as simple as possible, we want the largest such subspace.

Let $\bar{A}$ be the set of elements in F

$$\alpha(v + av', w) - \alpha(v, w) - a\,\alpha(v', w)$$
$$\alpha(v, w + aw') - \alpha(v, w) - a\,\alpha(v, w')$$

for $a \in \mathbb{R}$, $v, v' \in V$, $w, w' \in W$. Let A be the subspace generated by $\bar{A}$. You can easily check that $\bar{\mu}(a) = 0$ for any $a \in \bar{A}$ and, therefore, for any $a \in A$ since $\bar{\mu}$ is linear. Thus

natural insertion (mono)

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[r, "\beta"] \arrow[d, "\bar{\mu}"] & F/A \arrow[dl, "\bar{\bar{\mu}}"] \\
& Z &
\end{tikzcd}$$

$\beta : f \mapsto f + A$

$A \subset \operatorname{Ker} \bar{\mu}$

$\bar{\bar{\mu}}$ is unique

commutes for a unique $\bar{\bar{\mu}}$ and so $(\beta \circ \alpha, F/A)$ is a tensor product as Geroch has defined it. This is conventionally renamed

$$V \times W \xrightarrow{\otimes} V \otimes W$$

You can easily check that $\otimes$ is bilinear.

Geroch also shows that $V \otimes W$ is the free vector space on $S \times T$ where S is a basis of V and T is a basis of W. For example, $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$ for finite dimensional spaces.

Of course this applies to all vector spaces, so you can form, for example $V \otimes V^*$. Sometimes these come with "natural" mappings i.e.

$$(v, f) \mapsto f(v) \qquad\qquad V \times V^* \longrightarrow \mathbb{R}$$

This mapping is bilinear and so extends uniquely to $V \otimes V^* \to \mathbb{R}$. It's called a "contraction" and provides a map from, e.g.

$$A \otimes B \otimes V \otimes V^* \otimes C \longrightarrow A \otimes B \otimes C$$

In old-fashioned tensor terminology the dualed vector spaces are called "covariant" and the undualled spaces are called "contravariant".

For example, if $V = \mathbb{R}^4$, $(f_x : (x, y, z, t) \mapsto x) \in V^*$ etc.,

$$f_x \otimes f_x + f_y \otimes f_y + f_z \otimes f_z - f_t \otimes f_t \in V^* \otimes V^*$$

is the lorentz metric tensor. "Type $\binom{0}{2}$"

Here's another "natural" mapping. This time from $V \times V^* \to \mathrm{End}(V)$.

$$(v, f) \mapsto (v' \mapsto f(v')v)$$

[$\mathrm{End}(V) \equiv \mathrm{Mor}(V, V) \equiv \mathrm{Lin}(V; V)$, "endomorphisms"]

This one is also bilinear and so extends to a unique $V \otimes V^* \xrightarrow{\varphi} \mathrm{End}(V)$ as in

$$\begin{tikzcd} V \times V^* \arrow[r, "\otimes"] \arrow[dr] & V \otimes V^* \arrow[d, "\varphi"] \\ & \mathrm{End}(V). \end{tikzcd}$$

For example, pick some $v \otimes f + v' \otimes f' \in V \otimes V^*$. Then

$$\varphi(v \otimes f + v' \otimes f')(u) = \varphi(v \otimes f)(u) + \varphi(v' \otimes f')(u)$$

$= f(u)v + f'(u)v'$ using linearity of φ and the diagram.

This is often just abbreviated

$$(v \otimes f + v' \otimes f')(u) = f(u)v + f'(u)v'$$

leaving out the "φ" for convenience. Here's another example:

$\mathrm{End}(\mathbb{R}^2)$ includes rotations, e.g.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This can be written as the tensor

$$\cos\theta \; x \otimes f_x - \sin\theta \; y \otimes f_x + \sin\theta \; x \otimes f_y + \cos\theta \; y \otimes f_y \in V^* \otimes V$$

Is every $\psi \in End(V)$ a tensor in $V \otimes V^*$?

The answer is no, in general, but for finite dimensional vector spaces $End(V) \cong V \otimes V^*$. In any case, we can show that φ in the diagram above is a monomorphism. For practice, lets do this by putting in some explicit bases for V and V^*.

As usual, the convenient way to show that φ is mono is to show that $\mathrm{Ker}\,\varphi = \{0\}$, i.e. that $\varphi(x) = 0 \Rightarrow x = 0$.

Let

$$x = \sum_{jk}{}' a_{jk} v_j \otimes f_k \qquad \text{where} \quad \begin{array}{l} v_j \in B \text{ (some basis for } V) \\ f_k \in C \text{ (some basis for } V^*) \end{array}$$

$$\varphi(x) = \sum_{jk}{}' a_{jk}\, \varphi(v_j \otimes f_k) \qquad \text{since } \varphi \text{ is linear}$$

$$= \sum_{jk}{}' a_{jk}\, (v' \mapsto f_k(v') v_j) \qquad \text{using the diagram}$$

"0" in $End(V)$ is the map $(v' \mapsto 0)$, so $\varphi(x) = 0$

$$\Rightarrow \sum_{jk}{}' a_{jk}\, (v' \mapsto f_k(v') v_j) = (v' \mapsto 0) = (v' \mapsto \sum_{jk}{}' a_{jk} f_k(v') v_j)$$

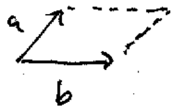
$$\Rightarrow \sum_{jk}{}' a_{jk} f_k(v') v_j = 0 \quad \text{for all } v' \in V$$

$$\Rightarrow \sum_{j}{}' \Big(\sum_{k}{}' a_{jk} f_k(v') \Big) v_j = 0 \Rightarrow \sum_{k}{}' a_{jk} f_k(v') = 0 \quad \text{for all } v', j$$

$$\Rightarrow \sum_{k}{}' a_{jk} f_k = (v' \mapsto 0) \Rightarrow a_{jk} = 0 \text{ for all } j, k \Rightarrow x = 0 \Rightarrow \varphi \text{ is mono.}$$

You can always put in the bases if you really want to.

One of the starting examples



area : $(a, b) \mapsto$ area of parallelopiped

is actually more than just bilinear. We also know that $\text{area}(x, x) = 0$ for all $x \in \mathbb{R}^2$ and this means that $\text{area}(a, b) = -\text{area}(b, a)$ [look at $\text{area}(a+b, a+b)$].

area is thus "alternating" as well as bilinear. This means that $\bar{\mu}$ zeros more elements of F than we thought and we can take the quotient with a bigger subspace and simplify things even further.

Let

$$\begin{tikzcd} V \times V \arrow[r, "\alpha"] \arrow[dr, "\mu"'] & F \arrow[r, "\beta"] \arrow[d, "\bar{\mu}"] & F/A \arrow[dl, "\bar{\bar{\mu}}"] \\ & Z & \end{tikzcd}$$

bilinear and alternating

$\bar{A} = \bar{A}$ from before $\cup$

$\alpha(v, v') + \alpha(v', v)$ etc.

A = subspace generated by the new, larger $\bar{A}$

$A \subset \text{Ker}\, \bar{\mu} \Rightarrow \bar{\bar{\mu}}$ exists and is unique.

In this case $(\beta \circ \alpha, F/A)$ is called a "wedge product" and is conventionally renamed

$$V \times V \xrightarrow{\wedge} V \wedge V$$

"$\wedge$" is both bilinear and alternating, so, $(v + av') \wedge v''$
$= v \wedge v'' + a v' \wedge v''$, $v \wedge v' = -v' \wedge v$, $v \wedge v = 0$ etc.

This has an obvious generalization to $V \wedge V \wedge V$ etc. where $v \wedge v' \wedge v''$ is alternating if any pair are swapped etc.

Wedge products are simpler than tensors and are <u>very</u> useful for, e.g. integration, since they represent volume elements.

For example suppose that V is n-dimensional and consider

$$\underbrace{V \wedge V \wedge V \wedge \cdots \wedge V}_{n \text{ times}} \equiv \text{"}\Lambda^n V\text{"}$$

Given an element of $\Lambda^n V$ like $v_1 \wedge v_2 \wedge \cdots \wedge v_n$, we can expand each of the v_j in some basis $\epsilon_1, \cdots, \epsilon_n$ of V. Thus, any element of $\Lambda^n V$ can be expressed as a linear combination of n-wedges of just the basis vectors. Since such a wedge is zero if any basis vector repeats, any element of $\Lambda^n V$ can be written as

$$t = K\, \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n \quad \text{for some } K \in \mathbb{R}$$

$$t \in \Lambda^n V$$

Notice that the mapping $t \mapsto K$ is an isomorphism [Proof: $K=0 \Rightarrow t=0$, so it's mono. It's also clearly epi].

$\Rightarrow$ $\Lambda^n V \cong \mathbb{R}$ it's a one dimensional vector space.

Now we get to do a very neat trick...

Given $\varphi \in \mathrm{End}(V)$, consider

$$v_1 \wedge v_2 \wedge \cdots \wedge v_n \longmapsto \varphi(v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n)$$

Since $\Lambda^n V$ is one dimensional,

$$\varphi(v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n) = K\, v_1 \wedge v_2 \wedge \cdots \wedge v_n$$

for some $K \in \mathbb{R}$. You can easily see that this same factor k occurs independent of $v_1 \wedge v_2 \wedge \cdots \wedge v_n$.

Proof. Any $w_1 \wedge w_2 \wedge \cdots \wedge w_n = a\, v_1 \wedge v_2 \wedge v_3 \wedge \cdots \wedge v_n$ for some $a \in \mathbb{R}$.
$\varphi(a v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n) = K a\, v_1 \wedge v_2 \wedge \cdots \wedge v_n = K\, w_1 \wedge w_2 \wedge \cdots \wedge w_n$.

K has a special name. It's called the <u>determinant</u> of φ.

Intuitively, you can see that $\det(\varphi)$ is just the factor by which the volume element of V changes under the mapping φ. In particular, $\det(\varphi) = 1$ mappings are volume preserving.

The properties of the determinant are now obvious, e.g.

$$\det(\psi \circ \varphi) = \det(\psi) \cdot \det(\varphi).$$

This means, by the way, that $\mathrm{Aut}(V) \xrightarrow{\det} \mathbb{R}^*$ is a group homomorphism from $\mathrm{Aut}(V)$ to the multiplicative group of nonzero reals. For example $\mathrm{Ker}(\det)$ is a volume preserving subgroup of $\mathrm{Aut}(V)$.

You can also easily show

Theorem: $\varphi \in \mathrm{End}(V)$ is an isomorphism iff $\det(\varphi) \neq 0$.

Proof. If φ is an isomorphism, $i_V = \varphi^{-1} \circ \varphi$, $\det(i_V) = 1 = \det(\varphi)\det(\varphi^{-1}) \Rightarrow \det(\varphi) \neq 0$.

Conversely, if φ is not an isomorphism, there must be a nonzero v in the kernel of φ. Choose a basis including v so that $v \wedge b_2 \wedge \cdots \wedge b_n$ is a basis of $\Lambda^n V$. But $\varphi(v) \wedge \varphi(b_2) \wedge \cdots \wedge \varphi(b_n) = 0 = \det\varphi \; v \wedge b_2 \wedge \cdots \wedge b_n$
$\Rightarrow \det\varphi = 0$.

Here's a sample application. The most general eigenvalue problem is this: Given $\varphi \in \mathrm{End}(V)$, find nonzero v_λ and $\lambda \in \mathbb{R}$ s.t.

$$\varphi(v_\lambda) = \lambda v_\lambda$$

However $\varphi(v_\lambda) = \lambda v_\lambda$ iff $(\varphi - \lambda i_V)(v_\lambda) = 0$. Since nonzero v_λ is in the kernel of $\varphi - \lambda i_V$, $\varphi - \lambda i_V$ is not an isomorphism, $\Rightarrow \det(\varphi - \lambda i_V) = 0$!