Topology (S.Y., Feb 2000).

A set X with "open" subsets

i) Containing both X and $\emptyset$

ii) Containing $\bigcup_\lambda \mathcal{O}_\lambda$ for open $\mathcal{O}_\lambda$ $(\lambda \in \Lambda)$.

iii) Containing $\mathcal{O} \cap \mathcal{O}'$ if $\mathcal{O}, \mathcal{O}'$ are open

is called a topological space. A particular collection of open sets on X is called a topology on X.

The objects in this category are topological spaces. The morphisms are continuous functions $X \xrightarrow{\varphi} Y$ i.e. functions satisfying $\varphi^{-1}[\mathcal{O}_y]$ is open for every open $\mathcal{O}_y \subset Y$.

monomorphism $\Leftrightarrow$ one-to-one continuous maps

epimorphism $\Leftrightarrow$ onto continuous map

isomorphism $\Leftrightarrow$ one-to-one, onto continuous map whose inverse is continuous.

This category has both direct sums and products, subobjects and quotients.

examples

$X, \emptyset$ only: The "indiscrete" topology

$\mathcal{P}(X)$: all subsets of X: The "discrete" topology

$\mathbb{R}$: open sets are subsets $\mathcal{O} \subset \mathbb{R}$ such that for any $x \in \mathcal{O}$, there is an $\epsilon > 0$ s.t. $|x' - x| < \epsilon \Rightarrow x' \in \mathcal{O}$.

Any metric space.

Any normed vector space.

$$\left.\begin{array}{l} n(v+v') \le n(v) + n(v') \\ n(av) = a\,n(v) \\ n(v) = 0 \text{ iff } v = 0 \end{array}\right\} \text{normed vector space}$$

Characterizing open sets

Theorem: A set $A \subset X$ is open iff for every $x \in A$, there is an open $\mathcal{O}_x$ such that $x \in \mathcal{O}_x \subset A$.

Proof. Suppose that A is open. Then for any $x \in A$, $A \subset A$. Conversely, if there are open $\mathcal{O}_x \ni x$, $\mathcal{O}_x \subset A$ for each $x \in A$, then $A = \bigcup_x \mathcal{O}_x$ is open.

Definitions:

Int (A) "interior" is the union of all open subsets of A.

Cl (A) "closure" is the intersection of closed supersets of A.

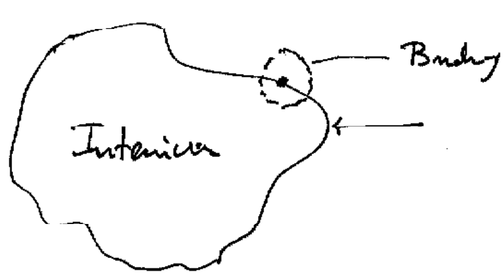
Bndry (A) "boundary" is $Cl(A) - Int(A)$.

Theorem (Geroch):

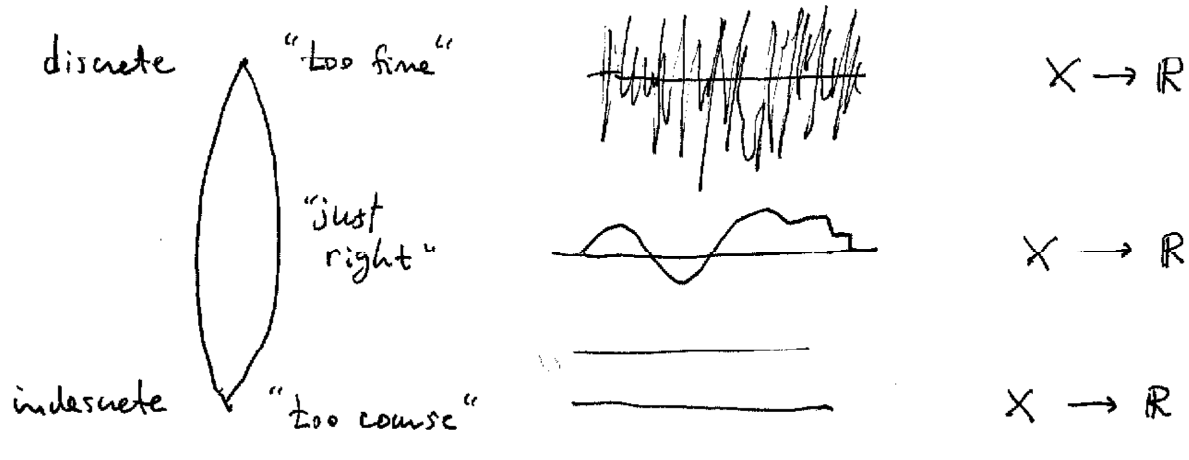
$x \in Int(A)$ iff some open neighborhood of x is a subset of A.

$x \in Cl(A)$ iff every open neighborhood of x intersects A.

$x \in Bndry(A)$ iff every open neighborhood of x intersects both A, A^c.



Consider topologies on a fixed set X. These are subsets of $\mathcal{P}(X)$, so we can order topologies by inclusion.



Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X. Then $\mathcal{T} \cap \mathcal{T}'$ is also a topology.

Proof, i) $\emptyset$ and X are both in $\mathcal{T} \cap \mathcal{T}'$

ii) $\bigcup_\lambda \mathcal{O}_\lambda$ is in $\mathcal{T} \cap \mathcal{T}'$ if $\mathcal{O}_\lambda$ are in $\mathcal{T} \cap \mathcal{T}'$.

iii) $\mathcal{O} \cap \mathcal{O}'$ is in $\mathcal{T} \cap \mathcal{T}'$ if $\mathcal{O}, \mathcal{O}'$ are in $\mathcal{T} \cap \mathcal{T}'$.

exercise: Extend this result to arbitrary intersections of topologies.

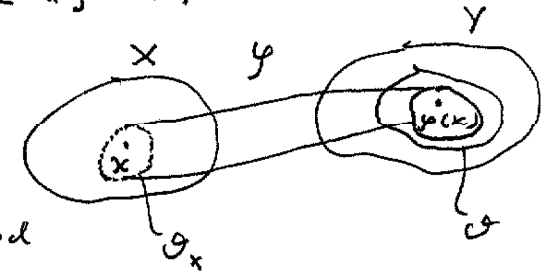
We can use this to define topologies generated by subsets. Let $\mathbb{A}$ be a collection of subsets of X. The topology <u>generated by</u> $\mathbb{A}$ is the intersection of all topologies containing $\mathbb{A}$ as ~~all open~~ open sets. Since the discrete topology contains $\mathbb{A}$, the intersection is not empty.

e.g. <u>Theorem</u>: The real line is the topology generated by open intervals.

Proof. Every open set in $\mathbb{R}$ is the union of open balls (intervals). The topology generated by intervals includes all such unions $\Rightarrow$ is $\mathbb{R}$.

What about morphisms?

Theorem. Let $\varphi : X \to Y$ be a mapping. φ is continuous iff for any $x \in X$ and any open neighborhood $\mathcal{O} \ni \varphi(x)$, there is an open $\mathcal{O}_x \ni x$ s.t. $\varphi[\mathcal{O}_x] \subset \mathcal{O}$.



Proof. Suppose that φ is continuous. Given any $x \in X$, $\mathcal{O} \ni \varphi(x)$ open in Y, clearly $\varphi^{-1}[\mathcal{O}]$ is an open neighborhood of x satisfying $\varphi[\varphi^{-1}[\mathcal{O}]] = \mathcal{O} \subset \mathcal{O}$.

Conversely, suppose that $\mathcal{O}$ is open in Y. Then, for any $x \in \varphi^{-1}[\mathcal{O}]$, there is an open $\mathcal{O}_x \ni x$ such that $\varphi[\mathcal{O}_x] \subset \mathcal{O}$. Since $\mathcal{O}_x \subset \varphi^{-1}[\mathcal{O}]$ for each x, $\varphi^{-1}[\mathcal{O}] = \bigcup_x \mathcal{O}_x$ is open.

We'll do more of this when we discuss "Nets".

Making topologies from existing topologies.

Let A be any subset of topological space X. I claim that the collection of sets $A \cap \mathcal{O}$ with $\mathcal{O}$ open in X is a topology on A.

Proof. i) Both $A = A \cap X$ and $\emptyset = A \cap \emptyset$ are open.

ii) If $A \cap \mathcal{O}_\lambda$ are open, then $\bigcup_\lambda (A \cap \mathcal{O}_\lambda) = A \cap (\bigcup_\lambda \mathcal{O}_\lambda)$ is also open.

iii) If $A \cap \mathcal{O}$ and $A \cap \mathcal{O}'$ are open, so ~~are~~ is $(A \cap \mathcal{O}) \cap (A \cap \mathcal{O}') = A \cap (\mathcal{O}' \cap \mathcal{O})$.

You can "induce" topologies with an arbitrary function.

$$X \xrightarrow{\varphi} Y$$

function

set

topological space

Let open sets in X be $\varphi^{-1}[\mathcal{O}_y]$ for open $\mathcal{O}_y$ in Y.

i) $X = \varphi^{-1}[Y]$ and $\emptyset = \varphi^{-1}[\emptyset]$ are open

ii) If $\varphi^{-1}[\mathcal{O}_\lambda]$ are open, then so is $\bigcup_\lambda \varphi^{-1}[\mathcal{O}_\lambda] = \varphi^{-1}[\bigcup_\lambda \mathcal{O}_\lambda]$.

iii) If $\varphi^{-1}[\mathcal{O}]$ and $\varphi^{-1}[\mathcal{O}']$ are open, then so is $\varphi^{-1}[\mathcal{O}] \cap \varphi^{-1}[\mathcal{O}'] = \varphi^{-1}[\mathcal{O} \cap \mathcal{O}']$.

exercise: show that a similar trick can be used to induce a topology on Y if X is a topological space.

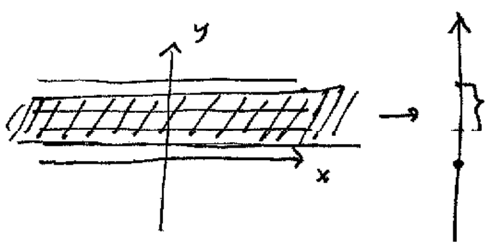
Quotient Topologies

Let X be a topological space with equivalence relation E. Then if

$$X \xrightarrow{\alpha} X/E \qquad \alpha : x \mapsto [x]$$

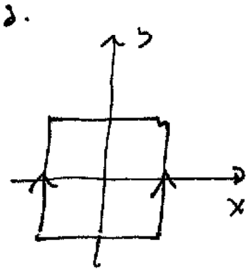
is the natural mapping, X/E with the topology induced by α is called the quotient topology.

ex: Let $(x,y)\,E\,(x',y')$ iff $x = x'$ be an equivalence relation on $\mathbb{R}^2$.

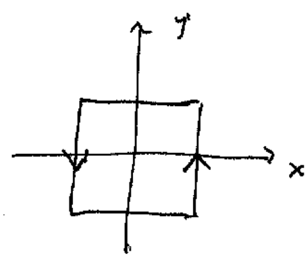


open iff the pullback into $\mathbb{R}^2$ is open.

e.g.



Torus



Möbius Strip.

Direct product topologies

Naturally, the first guess for the direct product of topological spaces X and Y is the cartesian product.

$$X \xleftarrow{\alpha} X \times Y \xrightarrow{\beta} Y \qquad \begin{array}{l} \alpha : (x,y) \mapsto x \\ \beta : (x,y) \mapsto y \end{array}$$

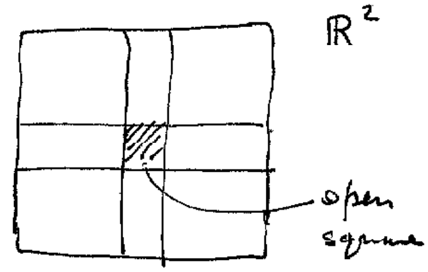
What topology do we put on $X \times Y$?

→ Since α, β are supposed to be morphisms, we definitely require $\alpha^{-1}[O_x]$ and $\beta^{-1}[O_y]$ to be open for open O_x, O_y. Try the topology generated by these sets only.

In general, the generated topology has open sets which are arbitrary unions of $\alpha^{-1}[O_x] \cap \beta^{-1}[O_y]$.

Consider

$$\begin{tikzcd} X & X \times Y \arrow[l, "\alpha"'] \arrow[r, "\beta"] & Y \\ & Z \arrow[u, "\gamma"] \arrow[ul, bend left, "\varphi"] \arrow[ur, bend right, "\psi"'] & \end{tikzcd}$$



From the set category, there is a unique γ that causes this to commute.. The only question is whether its continuous.

$$\gamma^{-1}[\alpha^{-1}[O_x] \cap \beta^{-1}[O_y]] = \gamma^{-1}[\alpha^{-1}[O_x]] \cap \gamma^{-1}[\beta^{-1}[O_y]]$$
$$= (\alpha \circ \gamma)^{-1}[O_x] \cap (\beta \circ \gamma)^{-1}[O_y] = \varphi^{-1}[O_x] \cap \psi^{-1}[O_y] \text{ is open.}$$

→ γ is continuous.

Any Questions....? I have one...

Why are unions of the form $\alpha^{-1}[O_x] \cap \beta^{-1}[O_y]$ sufficient?
This wasn't clear to me when I read Geroch, but you can think about it this way:

The general situation is having a mapping $\varphi : X \to Y$ where Y has a topology generated by some subsets $A \subset \mathcal{P}(Y)$. It would be convenient if the pullback of subsets in A implied that φ was continuous. However, this is not good enough because, roughly, φ is unconstrained in regions of Y not covered by a subset in A.

However, let's suppose that A includes Y and $\emptyset$ and that A is closed under intersection. Then it is easy to prove:

<u>Proposition</u>: Arbitrary unions of subsets in A is the topology generated by A.

Proof. First prove that we have a topology.

i) Y and $\emptyset$ are both in A by decree

ii) If U_α are arbitrary unions of elements in A, then $\bigcup_\alpha U_\alpha$ is also an arbitrary union.

iii) If $\bigcup_\alpha A_\alpha$ and $\bigcup_\beta B_\beta$ are open, then

$$\left(\bigcup_\alpha A_\alpha\right) \cap \left(\bigcup_\beta B_\beta\right) = \bigcup_\alpha \left(A_\alpha \cap \bigcup_\beta B_\beta\right) = \bigcup_\alpha \left(\bigcup_\beta \left(A_\alpha \cap B_\beta\right)\right)$$

is also an arbitrary union since $A_\alpha \cap B_\beta$ ~~are~~ in A.

Finally any topology generated from A must include all these open sets and therefore this is the topology generated by A.

Finally applying this to our direct product situation, notice that the collection of subsets

$$\alpha^{-1}[\mathcal{O}_x] \cap \beta^{-1}[\mathcal{O}_y] \quad \text{with } \mathcal{O}_x \text{ open in } X, \ \mathcal{O}_y \text{ open in } Y$$

includes $X \times Y$, $\emptyset$ and is closed under intersection. Therefore the open sets in the generated topology are arbitrary unions of these, just as Geroch ~~s~~ states.

Let's also polish off the direct sum.

$$X \xrightarrow{\alpha} X \cup_d Y \xleftarrow{\beta} Y \qquad \begin{array}{l} \alpha : x \mapsto (x, 1) \\ \beta : y \mapsto (y, 2) \end{array}$$

A subset $\mathcal{O}$ of $X \cup_d Y$ is <u>open</u> iff $\alpha^{-1}[\mathcal{O}]$ and $\beta^{-1}[\mathcal{O}]$ are both open. It's then easy to see that this is a topology. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, bend right, "\varphi"'] & X \cup_d Y \arrow[d, "\gamma"] & Y \arrow[l, "\beta"'] \arrow[dl, bend left, "\psi"] \\ & Z & \end{tikzcd}$$

As with the product, there is a unique mapping γ causing the diagram to commute (from the Category of Sets).

The only issue is whether γ ~~is~~ is continuous..

$$\alpha^{-1}[\gamma^{-1}[\mathcal{O}_z]] = (\gamma \circ \alpha)^{-1}[\mathcal{O}_z] = \varphi^{-1}[\mathcal{O}_z]$$

$$\beta^{-1}[\gamma^{-1}[\mathcal{O}_z]] = (\gamma \circ \alpha)^{-1}[\mathcal{O}_z] = \psi^{-1}[\mathcal{O}_z]$$

So $\gamma^{-1}[\mathcal{O}_z]$ is open.

Nets

A *directed set* is a partially ordered set Δ where for every δ, δ' in Δ, there is a $\delta'' \in \Delta$ such that $\delta \leq \delta''$ and $\delta' \leq \delta''$.

Given a topological space X, a *net* in X is a mapping $f : \Delta \to X$, typically denoted x_δ for $f(\delta)$.

A *net* x_δ *converges* to x if, for every open $\mathcal{O} \ni x$, there is a $\delta \in \Delta$ such that $\delta' \geq \delta \Rightarrow x_{\delta'} \in \mathcal{O}$.

Theorem: Let X be a Hausdorff topological space. Then if a net in X has a limit, it is unique.

Theorem: Let A be a subset of X. The closure of A is exactly the limit points of nets in A.

Proof. Suppose that x is a limit point of x_δ. Given any open neighborhood of x, there is an x_δ inside this neighborhood $\Rightarrow x$ is in the closure of A.

Conversely, suppose that x is in the closure of A. This means that every open neighborhood of x overlaps A. Let Δ be the directed set of such neighborhoods and for each $\mathcal{O} \in \Delta$, choose $x_\mathcal{O}$ in $A \cap \mathcal{O}$. Then, for any open neighborhood $\mathcal{O}$ of x, $\mathcal{O} \leq \mathcal{O}' \Rightarrow x_{\mathcal{O}'} \subset \mathcal{O} \Rightarrow x_\mathcal{O}$ converges to x.

You can see how fine-tuned these definitions are if you realize that the previous theorem is false if you use sequences instead of nets.

Compactness

A topological space is <u>compact</u> if every open cover has a finite subcover.

<u>examples</u>: The discrete topology on $\mathbb{R}$ is not compact because the cover consisting of individual points has no finite subcover.

The indiscrete topology is compact since every open cover must include the whole space.

$(0,1) \subset \mathbb{R}$ is not compact. For example, $(\frac{1}{n}, 1-\frac{1}{n})$ for $n \geq 3$ covers $(0,1)$ but has no finite subcover.

On the other hand, $[0,1]$ is compact (actually any closed, bounded subset of $\mathbb{R}^n$ is compact by the Heine-Borel theorem).

To understand why $(0,1)$ and $[0,1]$ are different, first consider sequences of ordered open or closed intervals

$$\mathcal{O}_1 \supset \mathcal{O}_2 \supset \mathcal{O}_3 \supset \cdots \quad \leftarrow \text{open intervals}$$

$$C_1 \supset C_2 \supset C_3 \supset \cdots \quad \leftarrow \text{closed intervals}$$

The ~~difference~~ key difference in these two situations is that $\bigcap_j \mathcal{O}_j$ may be empty, but $\bigcap_j C_j$ is nonempty. For example,

$$\bigcap_n (0, 1/n) = \emptyset \quad \text{but} \quad \bigcap_n [0, 1/n] = \{0\}.$$

To prove this for a general sequence $[l_1, h_1] \supset [l_2, h_2] \supset [l_3, h_3] \supset \cdots$, let h be the greatest lower bound of $h_1, h_2, \ldots$, l be the least upper bound of $l_1, l_2, \ldots$. Then $l \leq h$ and $[l, h] \subset \bigcap_j [l_j, h_j] \neq \emptyset$.

<u>Theorem.</u> Let $X \xrightarrow{\varphi} Y$ be a mapping. φ is continuous iff for every net x_δ converging to x in X, the net $\varphi(x_\delta)$ converges to $\varphi(x)$ in Y.

Proof. First suppose that φ is continuous. Choose any open $\mathcal{O} \ni \varphi(x)$. There is an open $\mathcal{O}_x \ni x$ s.t. $\varphi[\mathcal{O}_x] \subset \mathcal{O}$ and a δ s.t. $\delta' \geq \delta \Rightarrow x_{\delta'} \in \mathcal{O}_x$ $\Rightarrow \varphi(x_{\delta'}) \in \mathcal{O} \Rightarrow \varphi(x_\delta)$ converges to $\varphi(x)$.

Conversely, suppose that the condition is not satisfied but that φ is not continuous. Then there must be an $x \in X$, ~~such~~ open $\mathcal{O} \ni \varphi(x)$ such that $\varphi[\mathcal{O}_x] \not\subset \mathcal{O}$ for every open $\mathcal{O}_x \ni x$. For each such $\mathcal{O}_x$, choose $x_{\mathcal{O}_x} \in \varphi[\mathcal{O}_x] - \mathcal{O}$. Then $x_{\mathcal{O}_x}$ converges to x but $\varphi(x_{\mathcal{O}_x})$ does not converge to $\varphi(x)$.

~~One more definition. A net x_δ is said to have an <u>accumulation point</u> if, for any δ, there exists a $\delta' \geq \delta$ such that~~

A net x_δ is said to <u>accumulate</u> at x if, given any open $\mathcal{O}_x \ni x$, $\delta \in \Delta$, there exists a $\delta' \geq \delta$ s.t. $x_{\delta'} \in \mathcal{O}_x$.

Then we can argue, as Geroch does, that if $[0,1]$ is covered but has no finite ~~cover~~ subcover, then either $[0,\frac{1}{2}]$ or $[\frac{1}{2},1]$ ~~has~~ have no finite subcover. Iterating this argument we have an ordered sequence of closed subsets of $[0,1]$, $C_1 \supset C_2 \supset \cdots$ where none of the C_i have a finite subcover. This sequence must converge to some $a \in [0,1]$ which must be contained by some $\mathcal{O}_\lambda$ in the cover $\Rightarrow \Leftarrow$.

Theorem (Geroch): A topological space X is compact if and only if every net in X has an accumulation point.

Theorem: A closed subset of a compact topological space is compact.

Theorem: A compact subset of a Hausdorff space is closed.

Theorem: If $X \xrightarrow{\varphi} Y$ is continuous, C a compact subset of X, then $\varphi[C]$ is compact in Y.

Proof. Suppose $\mathcal{O}_\lambda$ covers $\varphi[C]$. Then $\varphi^{-1}[\mathcal{O}_\lambda]$ cover C.
A finite number of these $\varphi^{-1}[\mathcal{O}_1], \varphi^{-1}[\mathcal{O}_2], \ldots, \varphi^{-1}[\mathcal{O}_n]$ cover C.
$\Rightarrow \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ cover $\varphi[C]$.