S.Y., March 2000

Integration and Stokes theorem

Remember from last time ...

$$\Lambda^0(\mathcal{U}) \xrightarrow{d} \Lambda^1(\mathcal{U}) \xrightarrow{d} \Lambda^2(\mathcal{U}) \xrightarrow{d} \cdots \qquad \mathcal{U} \subset \mathbb{R}^n \quad \text{de Rham complex}$$

$$\omega = \sum_I{}' a_I(x)\, dx_I \equiv \sum_I{}' a_I(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

"differential k-form" on $\mathcal{U} \subset \mathbb{R}^n$.

$$d\omega = \sum_I{}' d\, a_I(x) \wedge dx_I \quad \text{exterior derivative} \quad d(d\omega) = 0$$

$$\begin{tikzcd} \Lambda^0(\mathcal{U}) \arrow[r, "d"] & \Lambda^1(\mathcal{U}) \arrow[r, "d"] & \cdots \\ \Lambda^0(V) \arrow[r, "d"] \arrow[u, "\varphi^*"] & \Lambda^1(V) \arrow[r, "d"] \arrow[u, "\varphi^*"] & \cdots \end{tikzcd}$$

Smooth $\mathcal{U} \xrightarrow{\varphi} V \subset \mathbb{R}^m$ induces pull backs of k-forms (Λ^* co-functor).

$$\varphi^*\omega(x) = \sum_I{}' (a_I \circ \varphi)(x)\, d_x(x_{i_1} \circ \varphi) \wedge d_x(x_{i_2} \circ \varphi) \wedge \cdots \wedge d_x(x_{i_k} \circ \varphi)$$

φ^* is uniquely determined by

a) $\Lambda^0(V) \xrightarrow{\varphi^*} \Lambda^0(\mathcal{U})$, $\varphi^* : f \mapsto f \circ \varphi$

b) The diagram commuting.

Refs: Differential Forms and Applications, Manfredo P. do Carmo
Differential Forms with Applications ..., Harley Flanders
Differential Forms in Algebraic Topology, Bott & Tu
Applied Differential Geometry, William Burke
Riemannian Geometry, Gallot, Hulin, Lafontaine

For example, let $\varphi : (0,\infty) \times [0, 2\pi) \to \mathbb{R}^2$ be given by $\varphi : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$

$\omega = dx \wedge dy \in \Lambda^2(\mathbb{R}^2)$ is the standard volume element.

$$\varphi^*\omega = d(x\circ\varphi) \wedge d(y\circ\varphi)$$

$$d(x\circ\varphi) = \frac{\partial(x\circ\varphi)}{\partial r}\,dr + \frac{\partial(x\circ\varphi)}{\partial\theta}\,d\theta = \cos\theta\,dr - r\sin\theta\,d\theta$$

$$d(y\circ\varphi) = \frac{\partial(y\circ\varphi)}{\partial r}\,dr + \frac{\partial(y\circ\varphi)}{\partial\theta}\,d\theta = \sin\theta\,dr + r\cos\theta\,d\theta$$

$$\varphi^*\omega = (\cos\theta\,dr - r\sin\theta\,d\theta)\wedge (\sin\theta\,dr + r\cos\theta\,d\theta)$$

$$\boxed{\text{Remember: } r : (r,\theta) \mapsto r \quad \theta : (r,\theta) \mapsto \theta}$$

$$= r\cos^2\theta\,dr\wedge d\theta - r\sin^2\theta\,d\theta\wedge dr = r\,dr\wedge d\theta$$

i.e. $\varphi^*\omega$ is the volume element in polar coordinates.

Example: $\omega \in \Lambda^3(\mathbb{R}^3)$ $\omega = dx\wedge dy \wedge dz$

$f : (r,\theta,\varphi) \mapsto (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$

$f^*(dx\wedge dy\wedge dz) = r^2\sin\theta\, dr\wedge d\theta\wedge d\varphi$ (exercise).

Remember the Laplacian in e.g. spherical coordinates from the back of Jackson?

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

this kind of thing is straight forward in de Rham theory.

Problem: What's the volume element in η, φ, z coordinates?

Example: Line integrals

Let $\omega = \sum_j a_j(x)\,dx_j$ be a 1-form on $\mathbb{R}^n$

We want to "integrate ω on a curve in $\mathbb{R}^n$". With $I = [a,b]$, let $I \xrightarrow{c} \mathbb{R}^n$ be a smooth curve.

Define $\int_c \omega = \int_I c^*\omega \equiv \int_a^b f(x)\,dx$ where $f\,dx = c^*\omega$.

By a "change of variables", we mean an isomorphism $I \xrightarrow{\varphi} I$.

<u>Exercise</u>: Prove that continuous $I \xrightarrow{\varphi} I$ is either increasing or decreasing.

The integral has this property:

If $I \xrightarrow{\varphi} I$ is increasing, $\int_c \omega = \int_{c\circ\varphi} \omega$.

<u>Exercise</u>: Prove this using the change of variables formula from calculus.

Remember that ω is "closed" if $d\omega = 0$, "exact" if $\omega = df$ for some $f \in \Lambda^0(\mathbb{R}^n)$. If ω is exact,

$$\int_c \omega = \int_c df = \int_I c^*(df) = \int_a^b d(f\circ c) = f(c(b)) - f(c(a))$$

$\Rightarrow \int_c \omega$ only depends on the end points.

$\Rightarrow \int_{loop} \omega = 0.$

Misc. Line integral results.

The following are equivalent

(a) ω is exact

(b) $\int_c \omega$ depends only on the end points

(c) $\int_c \omega = 0$ for all closed curves c

One can make contact with Homotopy theory from before

Theorem. If ω is a closed 1-form on $\mathcal{U} \subset \mathbb{R}^n$ and smooth $c, c' : [a,b] \to \mathcal{U}$ are homotopic, then $\int_c \omega = \int_{c'} \omega$.

Theorem. A closed 1-form on a simply connected set is exact.

<u>Integration of vector fields</u>

Recall that an inner product fixes an isomorphism

$$\alpha : v \mapsto (v' \mapsto \langle v, v' \rangle)$$

from $\mathbb{R}^n$ to $(\mathbb{R}^n)^* \equiv \Lambda^1(\mathbb{R}^n)$. We can use this to define integration of a vector field

$$\int_c v = \int_c \omega$$

where ω is the corresponding 1-form.

Integration of volume forms.

Let $\omega = f\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \in \Lambda^n(\mathbb{R}^n)$ where $f : \mathbb{R}^n \to \mathbb{R}$ has "compact support". Simply define

$$\int_{\mathbb{R}^n} f\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \equiv \int_{\mathbb{R}^n} f\, dx_1 dx_2 \cdots dx_n$$

Standard Riemann integral.

Notice that the left side depends on the order of the coordinates. In general, a "change of coordinates" means an isomorphism $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^n$. Let $\varphi_j \equiv x_j \circ \varphi$

$$d\varphi_1 \wedge d\varphi_2 \wedge \cdots \wedge d\varphi_n = dx_1(d\varphi) \wedge dx_2(d\varphi) \wedge \cdots \wedge dx_n(d\varphi)$$

$$= \det(d\varphi)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \qquad \det(d\varphi) \equiv J(\varphi) \text{ "Jacobian"}$$

$$\int_{\mathbb{R}^n} \varphi^* \omega = \int_{\mathbb{R}^n} (f \circ \varphi)\, J(\varphi)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

On the other hand, $\int_{\mathbb{R}^n} f\, dx_1 dx_2 \cdots dx_n = \int_{\mathbb{R}^n} (f \circ \varphi)\, |J(\varphi)|\, dx_1 dx_2 \cdots dx_n$

by the change of variable formula from calculus.

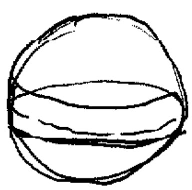
$$\Rightarrow \int_{\mathbb{R}^n} \varphi^* \omega = \pm \int_{\mathbb{R}^n} \omega \qquad \text{depending on } \operatorname{sgn} J(\varphi).$$

Integrals are only invariant under the orientation preserving subgroup of $\mathrm{Aut}(\mathbb{R}^n)$.

Definition. An atlas is <u>orientation preserving</u> if the Jacobians of all transition functions are positive.
A manifold is <u>orientable</u> if it has an ~~orientable~~ orientation preserving atlas.

e.g. The Möbius band is not orientable.

So far, we have defined integration of an n-form on $\mathbb{R}^n$. But on a manifold, the charts overlap



What to do?

A very handy trick is a <u>partition of unity</u>, a list $\rho_1, \rho_2, \dots, \rho_m$ of smooth functions from M to $[0,1]$ such that

(a) $\sum_j \rho_j(x) = 1$

(b) The support of each ρ_j is contained in a single chart.

Then we can define

$$\int_M \omega = \sum_{j=1}^{m} \int_M \rho_j \, \omega$$

You can prove:

a). A partition of unity exists for any atlas.

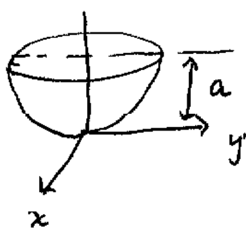
b) $\int_M \omega$ is independent of which partition of unity is used.

Stoke's Theorem

We've defined integration on manifolds. To get Stoke's famous theorem we have to introduce the idea of a "manifold with boundary".

e.g.

$$M = \{ (x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2 \text{ and } z \leq a \}$$



M is not a manifold because no point on the "boundary" (x, y, a) has an open neighborhood homeomorphic to $U \subset \mathbb{R}^2$.

Solution: A manifold with boundary is a manifold where the charts are isomorphic to the half space

$$H^n = \{ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0 \}$$

instead of to $\mathbb{R}^n$.

A point $p \in M$ is a boundary point if $\varphi_\alpha(p) = (0, x_2, x_3, \dots, x_n)$ for some $x_2, x_3, x \dots x_n \in \mathbb{R}$ where $(U_\alpha, \varphi_\alpha)$ is a chart with $p \in U_\alpha$.

It's actually not completely trivial, but you can prove that being a boundary point of M is not dependent on the atlas.

The boundary of M is denoted ∂M.

Theorem. The boundary ∂M of an n-manifold M is an $(n-1)$-manifold. If M is orientable, an orientation of M induces an orientation of ∂M.

Idea: Restricting $\varphi_\alpha : \mathcal{U}_\alpha \to H^n$
to $\mathcal{U}_\alpha \cap \varphi_\alpha^{-1}(\{(x_1, x_2, \dots, x_n) : x_1 = 0\})$
provides a chart around boundary point p mapping to
$\{(x_1, x_2, \dots, x_n) : x_1 = 0\} \cong \mathbb{R}^{n-1}$.

Stoke's Theorem

Let ω be an $(n-1)$-form on manifold M with boundary $\partial M \xrightarrow{i} M$. Then

$$\int_{\partial M} i^* \omega = \int_M d\omega \, .$$

Special cases:

Gauss' Law $(\omega \in \Lambda^2(\mathbb{R}^3))$

"Stoke's theorem" $(\omega \in \Lambda^1(\mathbb{R}^2))$

Fundamental theorem of Calculus $(\omega \in \Lambda^0(\mathbb{R}^1))$