Equivalence Relations

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A relation on a set X is just a subset of $X \times X$ [$X \times X$ is the "cartesian product" $X \times X = \{(x, x') : x, x' \in X\}$. The cartesian product is the direct product in the category of sets.]

Notation: If R is a relation on set X,

xRx' means (x, x') is in the relation R.

An equivalence relation is a relation with some extra properties. Relation E on X is an equivalence relation if

(a) xEx for all $x \in X$

(b) xEy implies yEx for all $x, y \in X$

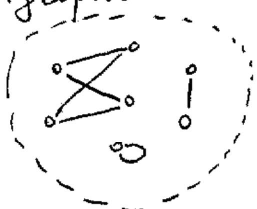
(c) xEy and yEz implies xEz for all $x, y, z \in X$.

More notation: $[x] \equiv \{x' \in X : x'Ex\}$

Such a subset of X is called an "equivalence class". The set of equivalence classes is ~~called~~ denoted X/E.

examples:

Bigraphs

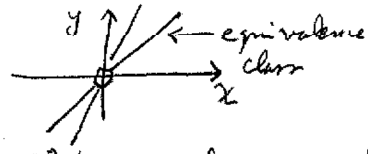
$X =$ 

x "is connected to" y is an equivalence relation.

X/E are the connected components of the Bigraph.

$X = \mathbb{R} \times \mathbb{R}$ ie. the x-y plane

let $(x, y) \sim (x', y')$ if $(x, y) = (ax', ay')$ for some nonzero $a \in \mathbb{R}$.



$\mathbb{R}^2/\sim$ are lines

$X = \mathbb{R}$

$x \sim x'$ if $x - x' = n2\pi$ is an equivalence relation.

$\mathbb{R}/\sim$ is isomorphic to the circle

From the category theoretic point of view, X/E is called a "Quotient". Here, X/E is a set and it's not that impressive. However in other categories, the analogous construction will produce quotient groups, quotient vector spaces, quotient topologies etc. In all these situations, there will be a construction like this:

Suppose E is an equivalence relation on X and $X \xrightarrow{f} Y$ "respects the structure of E" in the sense that $xEx' \Rightarrow f(x) = f(x')$. Let $\alpha : x \mapsto [x]$ be the "natural insertion" of $x \in X$ into its equivalence class $[x] \in X/E$. Then, there is a unique γ such that

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "f"'] & X/E \arrow[d, "\gamma"] \\ & Y \end{tikzcd}$$

$\alpha : x \mapsto [x]$ is obviously an epimorphism.

commutes.

Proof. Let $\gamma : \alpha(x) \mapsto f(x)$. We have to check that γ is a function. Since α is epi, we only have to check that $\alpha(x) = \alpha(x') \Rightarrow \gamma(x) = \gamma(x')$. $\Rightarrow \alpha(x) = \alpha(x') \Rightarrow [x] = [x']$ $\Rightarrow xEx' \Rightarrow f(x) = f(x') \Rightarrow \gamma(\alpha(x)) = \gamma(\alpha(x')) \Rightarrow \gamma$ is a function. Since α is epi, γ is unique because if some γ' also makes the diagram commute, then

$$\begin{tikzcd} X \arrow[r, "\alpha"] & X/E \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] & Y \end{tikzcd}$$ commutes $\Rightarrow \gamma = \gamma'$.

The Category of Groups   Geroch ch. 3, 4, 5, 6, 7   S.Y. Jan. 2000

Objects : Groups

Morphisms: "group homomorphisms"

i.e. $G \xrightarrow{\varphi} H$ where $\varphi(g g') = \varphi(g)\varphi(g')$ for $g, g' \in G$.

$[\Rightarrow \varphi(e_G) = e_H,\ \varphi(g^{-1}) = \varphi(g)^{-1}]$.

It's easy to check that this forms a category.

Geroch also shows that

monomorphism $\Leftrightarrow$ one-to-one group homomorphism

epimorphism $\Leftrightarrow$ onto group homomorphism

You can also easily show

isomorphism $\Leftrightarrow$ onto, one-to-one group homomorphism.

<u>examples</u>: $\mathbb{R}$eals and "+" are a group. $\mathbb{R}$ and "$*$" isn't quite (homework). $\mathbb{R}^2$ with vector addition is a group.

Suppose A is any object in any category, $\mathrm{Aut}(A) \equiv \{$isomorphisms from A to $A\}$ is a group with composition for group multiplication and i_A the identity.

Groups have both direct products and direct sums.
For a product, of G, H, we need an object "$G \times H$" and morphisms $G \xleftarrow{\alpha} G \times H \xrightarrow{\beta} H$ s.t. for any group Z, φ, ψ,

$$\begin{tikzcd}
G & G \times H \arrow[l, "\alpha"'] \arrow[r, "\beta"] & H \\
& Z \arrow[ul, bend left, "\varphi"] \arrow[u, "\gamma"'] \arrow[ur, bend right, "\psi"'] &
\end{tikzcd}$$

there is a unique γ causing the diagram to commute.

Let $G \times H$ be the cartesian product. We can make this into a group by defining

$$(g, h)(g', h) \equiv (gg', hh')$$

This is then a group with identity (e_G, e_H) and where the inverse of (g, h) is (g^{-1}, h^{-1}). It's then easy to guess that

$$\gamma : z \mapsto (\varphi(z), \psi(z))$$

and verify that this is the unique function causing the diagram to commute. Also, $\gamma(zz') = (\varphi(zz'), \psi(zz'))$ $= \gamma(z)\gamma(z')$, so we have proved that $(G \times H, \alpha, \beta)$ is a direct product of G and H.

Finding a direct sum is one of the problems...

Geroch Ch3 has two somewhat disorienting results:

Theorem: Every group is the subgroup of a permutation group.

The idea of the proof is to consider left multiplication

$$\psi_g : x \mapsto gx \qquad \psi_g : G \to G$$

This isn't necessarily a morphism, but it is easy to show that it is one-to-one and onto $\Rightarrow$ a permutation of G.

Geroch then shows that

$$g \mapsto \psi_g$$

is a group monomorphism from G to Perm(G).

The motivation for proving this is to ~~show that~~ find something not too complicated (Perm(G)) which contains all groups. I gather that math types like to do this kind of thing.

The second main result of Chapter 3 is showing that there is a free group on any set. This may also seem to you like a strange thing to do, but rest assured that you will see how important this kind of construction is in other categories.

~~then there's Theorem 10~~

Theorem 10. For any set S, there exists a free group on S.

$$\alpha : \begin{array}{l} a \mapsto \text{"}a\text{"} \\ b \mapsto \text{"}b\text{"} \\ \text{etc.} \end{array}$$

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & G \arrow[d, "\gamma"] \\ & H \end{tikzcd}$$

any function

unique group homomorphism

$S = \{a, b, c, \ldots\}$

$G \equiv \{\omega, a, aa, ab b' a a', \ldots \text{etc}\}$

remove all aa', $a'a$ etc. substrings

G is a group with concatenation as the group multiplication, with ω (the empty string) the identity and with $(ab'c)^{-1} = c'ba'$ etc.

These free constructions are actually very important. It will seem less arbitrary if you notice that free groups are unique.

Proof. Suppose $S \xrightarrow{\alpha} G$ and $S \xrightarrow{\alpha'} G'$ are free groups. Then

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & G \arrow[d, "\gamma"] \\ & G' \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & G \\ & G' \arrow[u, "\gamma'"'] \end{tikzcd}$$

commute for unique γ, γ'. $\Rightarrow \alpha' = \gamma \circ \alpha$, $\alpha = \gamma' \circ \alpha' \Rightarrow$

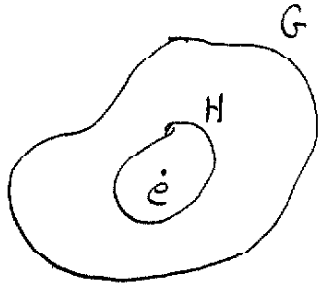
$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & G \arrow[d, "\gamma' \circ \gamma"] \\ & G \end{tikzcd}$$

commutes. But since

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & G \arrow[d, "i_G"] \\ & G \end{tikzcd}$$

also commutes, $\gamma' \circ \gamma = i_G$. Similarly, $\gamma \circ \gamma' = i_{G'} \Rightarrow G \cong G'$!

Subgroups



If subset H is closed under multiplication, closed under inversion and contains e, then H is also a group, called a <u>subgroup</u> of G.

Can we make "copies" of H?

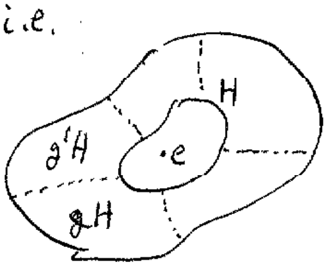
Try $gH \equiv \{gh : h \in H\}$ a "left coset" of H

then $eH = H = hH$ for any $h \in H$.

Theorem: Each $x \in G$ is in exactly one left coset of H.

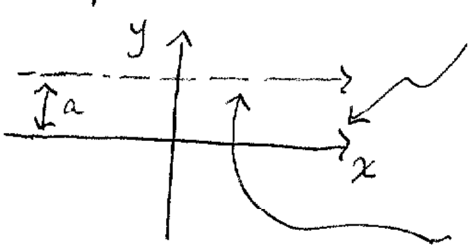
Proof. Clearly $x \in xH$. If x is also in gH, then $x = gh$ for some $h \in H$. $\Rightarrow xH = xh^{-1}H = gH$. QED.

i.e.



The natural mapping $H \to gH$ defined by $h \mapsto gh$ is an isomorphism of sets. $\Rightarrow$ All left cosets of H are isomorphic as sets.

For example, let $G = \mathbb{R}^2$ with vector addition as group multiplication.



$H = \{(x, 0) : x \in \mathbb{R}\}$ is a subgroup

$(0, a) + H$ is a horizontal line

$(0,a) + H = (56541.52, a) + H$

Note: cosets cover $\mathbb{R}^2$ without overlapping

cosets are all isomorphic as sets.

Let's try to make the left cosets of H into a group...

$$(gH)(g'H)$$

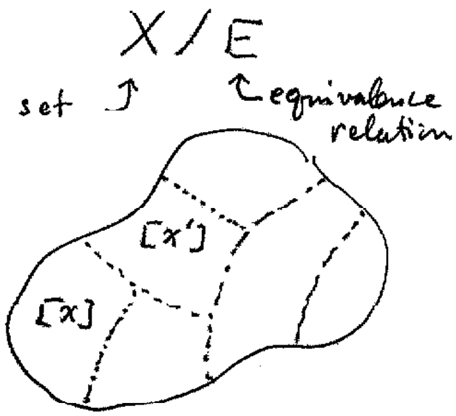
.... the problem is, this isn't necessarily a left coset.

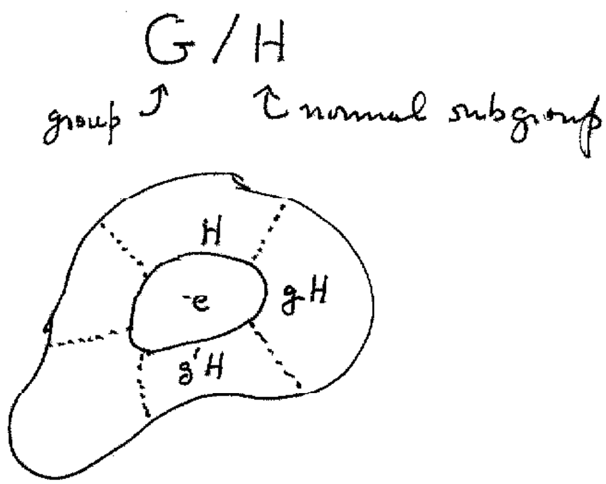
Suppose, then, that $gH = Hg$ for all $g \in G$ (such subgroups are called "Normal").

Then $gHg'H = gg'HH = gg'H$ = another coset

Now the set of left cosets is a group with identity eH and where $(gH)^{-1} = g^{-1}H$.

This is called the quotient group "G/H." Note the similarity to the category of sets.

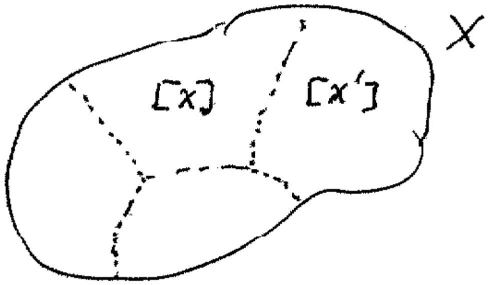




Normal subgroups. For any morphism $G \xrightarrow{\varphi} H$, $\operatorname{Ker} \varphi = \{ g \in G : \varphi(g) = e \}$ is a normal subgroup. Also $\operatorname{Im} \varphi$ is a subgroup of H.

For an example of a non-normal subgroup, consider the permutation group on $\{1, 2, 3\}$. Let g swap 1 and 2. Then $\{e, g\}$ is a subgroup but it's not normal.

In the category of sets, we found that an equivalence relation E on X causes X to be partitioned into equivalence classes



"X/E" is the sets of such equivalence classes.

If $X \xrightarrow{f} Y$ "respects" this structure in the sense that $f(x) = f(x')$ if x and x' are in the same class, then

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "f"'] & X/E \arrow[d, "\gamma"] \\ & Y \end{tikzcd} \qquad \alpha : x \mapsto [x]$$

f extends to a unique γ. This is the "main theorem" of equivalence clanes. Lets try to repeat this kind of argument for groups...

Let's try to repeat the "Main Quotient theorem"

Let $G \xrightarrow{\alpha} G/H$ be the "natural insertion" $\alpha: g \mapsto gH$. It's clearly an epimorphism.

Let's say that $G \xrightarrow{\psi} K$ is "respectful" of H if ~~if g and g' being in the same coset implies ψ~~ $\psi(g) = \psi(g')$ for any g, g' in the same coset.

Theorem: For any normal subgroup H of G, with $G \xrightarrow{\psi} K$ respectful of H, there is a unique monomorphism γ s.t.

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\psi"'] & G/H \arrow[d, "\gamma"] \\ & K \end{tikzcd}$$

commutes.

<u>Proof</u>. Let $\gamma: \alpha(g) \mapsto \psi(g)$. This clearly causes the diagram to commute. Since $\alpha(g) = \alpha(g') \Rightarrow gH = g'H$ $\Rightarrow g, g'$ are in the same coset $\Rightarrow \psi(g) = \psi(g')$, γ is a function. Since $\gamma(\alpha(g)\alpha(g')) = \gamma(\alpha(gg')) = \psi(gg')$ $= \psi(g)\psi(g') = \gamma(\alpha(g))\gamma(\alpha(g'))$, γ is also a group homomorphism. Since α is epi, γ is unique. To show that γ is mono, $\gamma(\alpha(g)) = \gamma(\alpha(g')) \Rightarrow \psi(g) = \psi(g')$ $\Rightarrow gH = g'H \Rightarrow \alpha(g) = \alpha(g') \Rightarrow \gamma$ is mono.

You can also easily show that $G \xrightarrow{\psi} K$ is respectful of H iff $H \subset \operatorname{Ker} \psi$. Proof: Suppose that ψ is respectful. Since e and $h \in H$ are in the same coset, $\psi(e) = \psi(h) \Rightarrow H \subset \operatorname{Ker} \psi$. Suppose $H \subset \operatorname{Ker} \psi$. If g, g' are in the same coset, then $gh = g'h'$ for some $h, h' \in H$ $\Rightarrow \psi(g) = \psi(g')$. $\Rightarrow \psi$ is respectful.

Suppose that we call $G \xrightarrow{\psi} K$ "exactly respectful" of normal subgroup H if

$\psi(g) = \psi(g')$ iff g and g' are in the same coset of H.

It's easy to show that $G \xrightarrow{\psi} K$ is exactly respectful iff $H = \mathrm{Ker}\,\psi$.

Proof.

A. Suppose $\mathrm{Ker}\,\psi = H$

g, g' in the same coset $\Rightarrow gh = g'h'$ for some $h, h' \in H$
$\Rightarrow \psi(g) = \psi(g')$.

$\psi(g) = \psi(g') \Rightarrow g g'^{-1} \in \mathrm{Ker}\,\psi \Rightarrow g = g'H \Rightarrow gH = g'H$
$\Rightarrow g, g'$ are in the same coset. $\Rightarrow \psi$ is exactly respectful.

B. Suppose $G \xrightarrow{\psi} K$ is exactly respectful.

e, h are in the same coset $\Rightarrow \psi(h) = e \Rightarrow H \subset \mathrm{Ker}\,\psi$.
Suppose $g \in \mathrm{Ker}\,\psi$. $\Rightarrow g, e$ are in the same coset $\Rightarrow g \in H$
$\Rightarrow \mathrm{Ker}\,\psi \subset H \Rightarrow \mathrm{Ker}\,\psi = H$.

One of the problems asks you to prove the neat result that $G \xrightarrow{\varphi} H$ is a ~~[illegible]~~ monomorphism iff $\mathrm{Ker}\,\varphi = \{e\}$. This is very handy.

Here's an example of how to use the quotient theorem.

Let $\mathbb{R}^*$ be the multiplicative group of nonzero reals. $\{+1,-1\}$ is a subgroup of this (normal).

What is $\mathbb{R}^*/\{+1,-1\}$?

Consider

$$\begin{tikzcd} \mathbb{R}^* \arrow[r, "\alpha"] \arrow[dr, "\mathrm{abs}"'] & \mathbb{R}^*/\{+1,-1\} \arrow[d, "\gamma"] \\ & \mathbb{R}^+ \end{tikzcd}$$

$\{+1,-1\} = \mathrm{Ker}(\mathrm{abs})$ i.e. abs is exactly respectful of $\{+1,-1\}$.

multiplicative group of positive reals.

$\mathrm{abs}: r \mapsto |r|$ is a morphism from $\mathbb{R}^*$ to $\mathbb{R}^+$ because $\mathrm{abs}(rr') = \mathrm{abs}(r)\,\mathrm{abs}(r')$. We already know from the theorem that the diagram commutes for a unique mono γ. Since abs is epi, γ is also epi $\Rightarrow \gamma$ is an isomorphism $\Rightarrow \mathbb{R}^*/\{+1,-1\} \cong \mathbb{R}^+$.

Isn't that cute? This idea is of help in the homework also.