S. Y. March 2000

Misc. Manifold Items; Lie Groups

Correction from last time

$$(r, \vec{A}) \cdot (r', \vec{A}') = (r r' - \vec{A} \cdot \vec{A}', r \vec{A}' + r' \vec{A} + \vec{A} \times \vec{A}')$$ is correct.

I had a prime misplaced.

Before going to Lie groups, let's think about manifolds a bit more. For example, we've seen that for p in three-manifold M, we have

$$T(M,p) \cong \mathbb{R}^3 \qquad T(M,p)^* \cong (\mathbb{R}^3)^*$$

$$\{\epsilon_x, \epsilon_y, \epsilon_z\} \qquad \{dx, dy, dz\} \quad \leftarrow \text{bases}$$

Since 3 is finite $T(M,p)^{**} \cong T(M,p)$ $(\equiv V)$. For example,

$$V^* \xrightarrow{\frac{\partial}{\partial x}} \mathbb{R} \text{ defined by } \frac{\partial}{\partial x} : f \mapsto \left.\frac{\partial f}{\partial x}\right|_p \text{ is in } T(M,p)^{**}$$

You can easily see that $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ are independent and therefore are a basis for $T(M,p)^{**}$.

$T(M,p)$ and $T(M,p)^{**}$ are called the "tangent space"

$T(M,p)^*$ is called the "cotangent space".

Thus we see that vector fields are equivalent to first order differential operators on a manifold like

$$X = \sum_{j=1}^{3} a_j(p) \frac{\partial}{\partial x_j}$$

Since a vector field like X above is a differential operator, we can define it's action on smooth elements of $\Lambda^0(M)$:

$$X(f) = \sum_{j=1}^{n} a_j(p) \frac{\partial f}{\partial x_j}\Big|_p \in \Lambda^0(M)$$

Notice, however that

$$YX(fg) = Y(f)X(g) + fYX(g) + Y(g)X(f) + gYXf$$

does not satisfy Liebnitz's rule and therefore isn't a first order differential operator. However, you can easily check that the <u>Lie bracket</u>

$$[X,Y] \equiv XY - YX$$

Does satisfy $[X,Y](fg) = f[X,Y]g + g[X,Y]f$ and is, therefore, a vector field.

Notice, incidentally that

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Given a vector field X (i.e. a section of $TM \xrightarrow{\pi} M$), we can define an <u>integral curve</u> $I \xrightarrow{c} M$ defined to have the property

$$c'(t) = X(c(t))$$

Theorem. For any point p in manifold M, there exists an open interval $I_p \ni 0$ and smooth $I_p \xrightarrow{c_p} M$ such that $c_p'(t) = X(c_p(t))$. Given I_p, c_p is unique.

This is a consequence of the basic existence and uniqueness theorem of the theory of ODEs.

The map $\tilde{\mathcal{F}}_X^t : p \mapsto c_p(t)$ is called the "flow" of a vector field.

It can be proved that within some open $\mathcal{O} \ni p$, the flow has the group property

$$\mathcal{F}_X^{t+t'} = \tilde{\mathcal{F}}_X^{t'} \circ \mathcal{F}_X^t$$

Example: $F = ma$, $F = -\nabla V$

$$m \frac{d^2 x}{dt^2} = -\frac{\partial V}{\partial x} \qquad q \equiv x \qquad p \equiv m \frac{dx}{dt}$$

$$\frac{dq}{dt} = \frac{p}{m} \qquad \frac{dp}{dt} = -\frac{\partial V}{\partial q}$$ are Hamilton's equations.

$$\mathcal{E}(p, q) = \frac{1}{2m} p^2 + V(q)$$

$$\frac{dq_j}{dt} = \frac{\partial \mathcal{E}}{\partial p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\partial \mathcal{E}}{\partial q_j}$$

In general, we are looking for the integral curves of the vector field

$$H = \sum_{j=1}^{n} \left(\frac{\partial \mathcal{E}}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial \mathcal{E}}{\partial q_j} \frac{\partial}{\partial p_j} \right)$$

a "Hamiltonian" vector field.

The flow of H preserves the "symplectic form"

$$\sigma = \sum_{j=1}^{n} dp_j \wedge dq_j$$

Since $\underbrace{\sigma \wedge \sigma \wedge \cdots \wedge \sigma}_{n}$ is the volume form, this is also preserved by the flow. This is Liouville's theorem.

Lie Groups

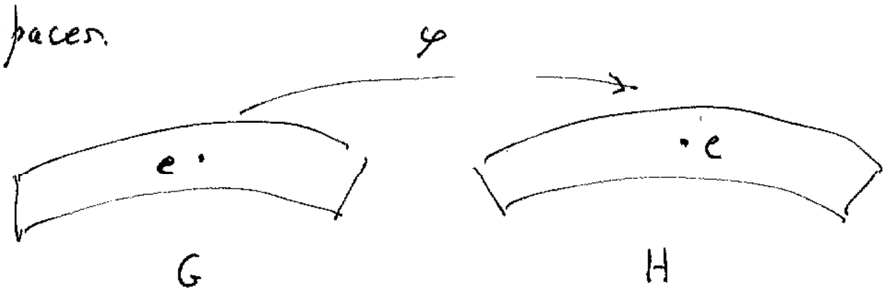
A Lie group is a group which is also a manifold where $(g, g') \mapsto gg'$ and $g \mapsto g^{-1}$ are smooth. Morphisms in this category are smooth maps which are also group homomorphisms.

Examples:

$\mathbb{R}^n$, S^1, S^3, $SO(n)$, $O(n)$, $SU(n)$, $Sp(n)$

the Lorentz group, E_7 are all Lie groups

Unlike manifolds in general, Lie groups are almost totally understood. Because every Lie morphism $G \xrightarrow{\varphi} H$ preserves the identity T_e is a functor from Lie groups to normed finite dimensional vector spaces.

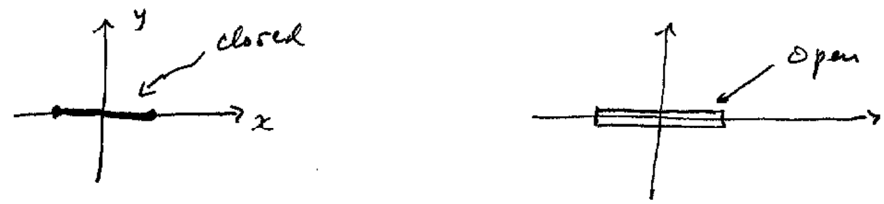


The main insight about Lie groups is this:

> Any open neighborhood of the identity generates the whole group.

This means that the action of any morphism $G \xrightarrow{\varphi} H$ is determined by what it does in any neighborhood of $e \in G$. We thus expect $G \xrightarrow{\varphi} H$ to be determined by $T_eG \xrightarrow{T_e\varphi} T_eH$.

For example, $\mathbb{R}^2, +$ is a Lie group. We claim that any open neighborhood of 0 generates $\mathbb{R}^2$. A closed neighborhood will not do, e.g.



Theorem: Let G be a connected Lie group. Then any neighborhood of the identity generates G.

Proof. Let $\mathcal{O} \ni e$ be open and H be any subgroup of G containing $\mathcal{O}$. Let $L_g : g' \mapsto gg'$ be left multiplication, a smooth map. $H = \bigcup_{h \in H} L_h[\mathcal{O}] \Rightarrow H$ is open. Also $H^c = \bigcup_{\bar{h} \in H^c} L_{\bar{h}}[\mathcal{O}] \Rightarrow H^c$ is open $\Rightarrow H$ is closed $\Rightarrow H$ is connected $\Rightarrow H = G$. $\Rightarrow$ The subgroup generated by $\mathcal{O}$ is G.

$$G \xrightarrow{\varphi} H$$

$$T_e G \xrightarrow{T_e \varphi} T_e H$$

Is every linear map $T_e G \longrightarrow T_e H$ the differential of a homomorphism?

We expect $T_e : \varphi \mapsto T_e \varphi$ to be one-to-one but

To find out that the answer is "no", we look for a condition on $T_e \varphi$ that φ is a morphism.

$$\varphi(g g') = \varphi(g)\varphi(g')$$

is the same statement as

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "L_g"'] & H \arrow[d, "L_{\varphi(g)}"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commuting. However

Also

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\Psi_g"'] & H \arrow[d, "\Psi_{\varphi(g)}"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes. Let's work with this one since $\Psi_g : g' \mapsto g g' g^{-1}$ fixes e and we are trying to stay in the neighborhood of e.

Applying the T_e functor, we have

$$\begin{tikzcd} T_e G \arrow[r, "T_e \varphi"] \arrow[d, "T_e \Psi_g"'] & T_e H \arrow[d, "T_e \Psi_{\varphi(g)}"] \\ T_e G \arrow[r, "T_e \varphi"] & T_e H \end{tikzcd}$$

Let $Ad : G \to Aut(T_e G)$ be
$Ad : g \mapsto T_e \Psi_g$
"adjoint"

$$\Rightarrow \quad Ad(\varphi(g))(T_e \varphi(v)) = T_e \varphi((Ad(g))(v)) \qquad v \in T_e G, \; g \in G$$

There is thus a commuting diagram for each $g \in G$. To get a constraint purely in the tangent spaces, fix a $v \in T_e G$. Then the condition above is

$$\begin{tikzcd} G \arrow[r, "g \mapsto Ad(g)(v)"] \arrow[d, "\varphi"'] & T_eG \arrow[d, "T_e\varphi"] \\ H \arrow[r, "h \mapsto Ad(h)(T_e\varphi(v))"] & T_eH \end{tikzcd}$$

commuting.

We can get a condition only involving the tangent spaces by applying T_e one more time.

Let $ad(v) = T_e(g \mapsto Ad(g)(v))$

$$\begin{tikzcd} T_eG \arrow[r, "ad(v)"] \arrow[d, "T_e\varphi"'] & T_eG \arrow[d, "T_e\varphi"] \\ T_eH \arrow[r, "ad(T_e\varphi(v))"] & T_eH \end{tikzcd}$$

i.e. for every $v' \in T_eG$,

$$\cancel{T_e\varphi(ad(v)(v')) = [ad(T_e(\varphi(v)))(T_e\varphi(v')]}$$

$$T_e\varphi(ad(v)(v')) = ad(T_e\varphi(v))(T_e\varphi(v'))$$

$[x,y] \equiv ad(x)(y)$ is customary.

$$T_e\varphi([v,v']) = [T_e\varphi(v), T_e\varphi(v')]$$

$\Rightarrow$ not all linear maps will do. Only those which preserve the bracket can be differentials of Lie group homomorphisms.

For example, let $G = \mathrm{Aut}(\mathbb{R}^3)$ in the vector space category. This is a Lie group.

$$\psi_L : L' \mapsto L \circ L' \circ L^{-1} \quad G \to G \text{ is "conjugation"}$$

$$\psi_L(L'+H) = \psi_L(L') + \psi_L(H) + 0 \Rightarrow T_e \psi_L = \psi_L$$

$\Rightarrow \mathrm{Ad} : L \to \psi_L$ is the adjoint as we have defined it.

$$\mathrm{ad}(M) \equiv T_e(L \to \mathrm{Ad}(L)(M))$$

$$= T_e(L \to L \circ M \circ L^{-1}) \equiv T_e(\delta)$$

$$\delta(L+H) = (L+H) \circ M \circ (L+H)^{-1}$$

$$= L \circ M \circ L^{-1} + H \circ M \circ L^{-1} - L \circ M \circ H + \mathcal{O}(H^2)$$

$$\delta(1+H) = M + H \circ M - M \circ H + \mathcal{O}(H^2)$$

$$\mathrm{ad}(M)(H) = H \circ M - M \circ H$$

As a matrix group, for example, the bracket is the standard commutator.

Theorem. Let G, H be Lie groups with G connected and simply connected. A map $T_eG \to T_eH$ is the differential of a homomorphism iff it preserves the bracket.

T_eG with $[\,]$ is a Lie Algebra.

A Lie Algebra is a vector space with a bilinear antisymmetric product $[\,] : V \times V \to V$ satisfying

$\boxed{\text{See Geroch Ch 19}}$

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Examples:

Let G be the subgroup of $\mathrm{Aut}(\mathbb{R}^n)$ preserving the euclidean inner product $\langle\,\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

For any $X \in T_e G$, we can find a curve $c : I \to G$ such that $c'(0) = X$. Then

$$\langle c(t)(v), c(t)(v') \rangle = \langle v, v' \rangle \quad \text{for all } v, v' \in \mathbb{R}^n$$

Taking the derivative and evaluating at $t=0$,

$$\langle Xv, v' \rangle + \langle v, Xv' \rangle = 0$$

$T_e G$ is the vector space of maps satisfying this condition. In terms of matrices, X is an anti-symmetric $n \times n$ matrix.

Let G be the subgroup of $\mathrm{Aut}(\mathbb{R}^n)$ with determinant 1. Similarly, for basis ϵ_i of $\mathbb{R}^n$,

$$c(t)\epsilon_1 \wedge c(t)\epsilon_2 \wedge \cdots \wedge c(t) \wedge \epsilon_n = \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n$$

$$\Rightarrow \sum_j{}' \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge X\epsilon_j \wedge \cdots = \mathrm{trace}(X)\, \epsilon_1 \wedge \epsilon_2 \wedge \cdots \wedge \epsilon_n = 0$$

$\Rightarrow$ $T_e G$ is the vector space of traceless linear maps. In matrix terms, it is the vector space of $n \times n$ traceless matrices.

The Exponential map

For each $X \in T_e G$, we can construct a vector field by defining

$$v_X(g) = T_e L_g(X)$$

Let $\varphi_X : \mathbb{R} \to G$ be the corresponding integral curve.

You can show that φ_X is a Lie group homomorphism with tangent X at 0. "One parameter subgroup".

this uniquely determines φ_X.

Define $\exp : T_e G \to G$ by

$$\exp : X \mapsto \varphi_X(1)$$

The exponentials cover an open neighborhood of the identity and thus determine the action of all morphisms + generate G.

$$\begin{tikzcd} G \arrow[r, "\varphi"] & H \\ T_e G \arrow[u, "\exp"] \arrow[r, "T_e \varphi"'] & T_e H \arrow[u, "\exp"'] \end{tikzcd}$$

commutes for any Lie group morphism φ.

In the case GL_n,

$$\exp : X \mapsto 1 + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

is the unique map taking $0 \to 1$ whose differential at the origin is the identity. It is inverted by $\exp(-X)$. $\exp(aX)$ is a one parameter subgroup of GL_n.

For any Lie group, there is a neighborhood $\mathcal{U} \subset T_e G$ such that

$$\exp(X)\exp(Y) = \exp(C(X,Y))$$

for some fixed $C : \mathcal{U} \times \mathcal{U} \to T_e G$.

This is the "Campbell - Baker - Hausdorff" formula (Derived in closed form by Dynkin!).

In the case of GL_n

$$C(X,Y) = X + Y + \frac{1}{2}[X,Y] + \cdots$$

For every Lie group, there is a unique (up to a constant) left-invariant volume form. Left invariance means

$$L_g^* \omega = \omega$$

So, for example,

$$\cancel{\int_G f \omega} \quad \int_G L_g^*(f\omega) = \int_G (f \circ L_g)\, L_g^* \omega = \int_G (f \circ L_g)\, \omega$$

Usually, the left invariant volume and the right invariant volume are the same. This is "Haar measure" (See Geroch Ch. 43 for measure).

For example, when G is the symmetry group of a probability problem, it gives the unique prior (measure) which is G invariant.