(S.Y. March 2000)

The "main quotient theorem" for topological spaces.

Remember that if we have a topological space X with equivalence relation E, we define the quotient topology on X/E by

$$X \xrightarrow{\alpha} X/E \qquad \alpha : x \mapsto [x]$$

simply declaring all subsets of X/E open if their pullback through α is open.

Suppose that we have continuous $X \xrightarrow{\varphi} Y$ which "respects" E, i.e. $xEx' \Rightarrow \varphi(x) = \varphi(x')$. Then from the category of sets, we know that

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & X/E \arrow[d, "\gamma"] \\ & Y \end{tikzcd}$$

commutes for a unique function γ. To show that γ is continuous, simply apply the powerset/pullback cofunctor.

$$\begin{tikzcd} \mathcal{P}(X) & \mathcal{P}(X/E) \arrow[l, "\alpha^{-1}"'] \\ & \mathcal{P}(Y) \arrow[u, "\gamma^{-1}"'] \arrow[ul, "\varphi^{-1}"] \end{tikzcd}$$

For $\mathcal{O}$ open in Y, $\gamma^{-1}[\mathcal{O}]$ has the property that $\alpha^{-1}[\gamma^{-1}[\mathcal{O}]]$ is equal to $\varphi^{-1}[\mathcal{O}] \Rightarrow$ is open. i.e. $\gamma^{-1}[\mathcal{O}]$ is open $\Rightarrow \gamma$ is continuous.

Notes on Homotopy and Covering Spaces

S.Y.
March 2000

Is having a hole a topological invariant?

$I = [0,1]$ with the usual topology.

A <u>loop</u> with base point x in topological space X is a morphism $I \xrightarrow{\alpha} X$ with $\alpha(0) = \alpha(1) = x$. Two loops are <u>homotopic</u> if they can be continuously interpolated, i.e. α, α' are homotopic if there is some continuous

$$i : I \times I \to X$$

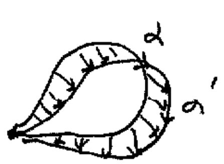
with $i(0,t) = \alpha(t)$, $i(1,t) = \alpha'(t)$.

For example, given a loop α, we can <u>reparameterize</u> it by composing with some continuous $p : I \to I$ with $p(0) = 0$, $p(1) = 1$. Then $\alpha \circ p$ is also a loop with base point x and α and $\alpha \circ p$ are homotopic because

$$i : (s,t) \mapsto \alpha(p(t) + s(t - p(t)))$$

continuously interpolates between α and α'.

Next, notice that "homotopic" is an equivalence relation which we will call H.



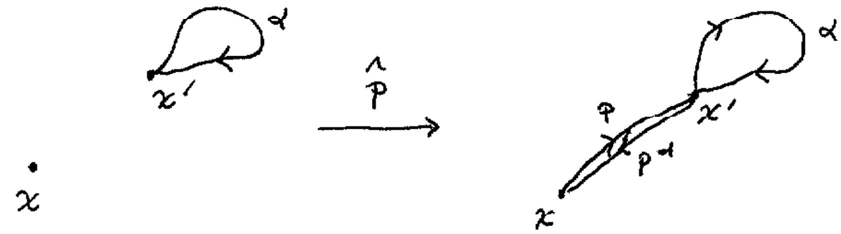
$\alpha H \alpha$

$\alpha H \alpha' \Rightarrow \alpha' H \alpha$

$\alpha H \alpha'$ and $\alpha' H \alpha'' \Rightarrow \alpha H \alpha''$

See Geroch for the proof.

Let $p : I \to X$ have $p(0) = x$, $p(1) = x'$ and let $\hat{p}$ ~~take~~ have the following action

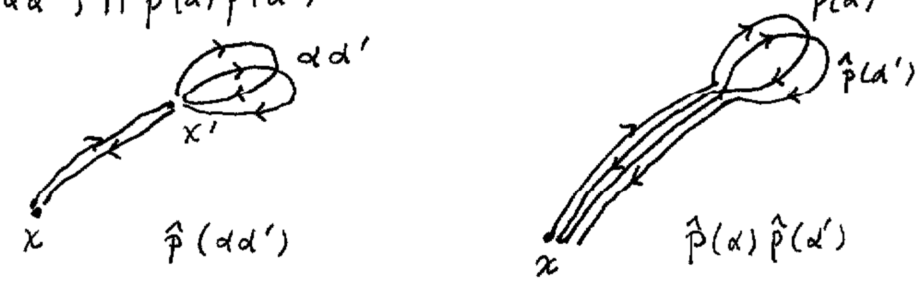


i.e. $\hat{\hat{p}} : [\alpha] \mapsto [\hat{p}(\alpha)]$ goes from $\pi_1(X, x')$ to $\pi_1(X, x)$.

Note that $\hat{\hat{p}}$ is a function because $[\alpha] = [\alpha'] \Rightarrow \alpha H \alpha' \Rightarrow \hat{p}(\alpha) H \hat{p}(\alpha')$
$\Rightarrow [\hat{p}(\alpha)] = [\hat{p}(\alpha')]$.

$\hat{p}$ also interacts nicely with the loop product:

$\hat{p}(\alpha\alpha') \, H \, \hat{p}(\alpha)\hat{p}(\alpha')$



$\Rightarrow \hat{\hat{p}}([\alpha][\alpha']) = \hat{\hat{p}}([\alpha\alpha']) = [\hat{p}(\alpha\alpha')] = [\hat{p}(\alpha)\hat{p}(\alpha')] = [\hat{p}(\alpha)][\hat{p}(\alpha')]$
$= \hat{\hat{p}}([\alpha])\hat{\hat{p}}([\alpha'])$ $\Rightarrow$ $\hat{\hat{p}}$ is a group homomorphism. You can also easily guess that $\hat{\hat{p}}$ is invertible. $\Rightarrow \pi_1(X, x) \cong \pi_1(X, x')$.

Given a loop α in X, and continuous $X \xrightarrow{\varphi} Y$, we can send the loop into Y via

$$
\begin{tikzcd}
X \arrow[r, "\varphi"] & Y \\
I \arrow[u, "\alpha"] \arrow[ur, "\varphi\circ\alpha"'] &
\end{tikzcd}
$$

$\alpha(0) = x$,

$\varphi\circ\alpha$ also a loop. $[\varphi\circ\alpha] \in \pi_1(Y, \varphi(x))$

You can easily see that the function $\Psi : \alpha \mapsto \varphi\circ\alpha$ is respectful of homotopy because if $i : I\times I \to X$ interpolates between α and α', then $\varphi\circ i$ interpolates between $\Psi(\alpha)$ and $\Psi(\alpha')$.

Also, notice that

$$\varphi\circ(\alpha\alpha') = \varphi\circ\left(t \mapsto \begin{matrix}\alpha(t) \text{ if } t \le 1/2 \\ \alpha'(2(t-1/2)) \text{ if } t \ge 1/2\end{matrix}\right) = \left(t \mapsto \begin{matrix}\varphi(\alpha(t)) \text{ if } t \le 1/2 \\ \varphi(\alpha'(2(t-1/2))) \text{ if } t \ge 1/2\end{matrix}\right)$$

$$= (\varphi\circ\alpha)(\varphi\circ\alpha').$$

This means that $\lambda : [\alpha] \mapsto [\varphi\circ\alpha]$ is a group homomorphism because $\lambda([\alpha][\alpha']) = \lambda([\alpha\alpha']) = [\cancel{\alpha}\,\varphi\circ(\alpha\alpha')] = [(\varphi\circ\alpha)(\varphi\circ\alpha')]$

$$= [(\varphi\circ\alpha)][(\varphi\circ\alpha')] = \lambda([\alpha])\,\lambda([\alpha']).$$

Let's organize these results as a functor.

Introduce the "category of pointed topological spaces":

Objects: pairs (X, x) where X is a topological space and $x \in X$.

Morphisms: A morphism from (X,x) to (Y,y) is a continuous map φ such that $\varphi(x) = y$.

~~exercise~~
exercise: prove that this is a category.

$$(X, x) \xrightarrow{\varphi} (Y, y)$$

π_1 as a functor

$$\pi_1(X, x) \xrightarrow{\pi_1(\varphi)} \pi_1(Y, y)$$

where $\pi_1(\varphi) \equiv [\alpha] \mapsto [\varphi \circ \alpha]$

exercise: Prove that π_1, as defined, is a functor.

Since functors preserve commuting diagrams, they take isomorphisms to isomorphisms. Thus, if $(X,x) \cong (Y,y)$, then $\pi_1(X,x) \cong \pi_1(X,y) \Rightarrow$ you can't get rid of a hole continuously.

Definition: If $\pi_1(X,x) \cong \{e\}$ and X is path connected, then X is called "simply connected".

e.g. The 2-plane with n holes, then $\pi_1(X,x) \cong$ Free group on n elements.

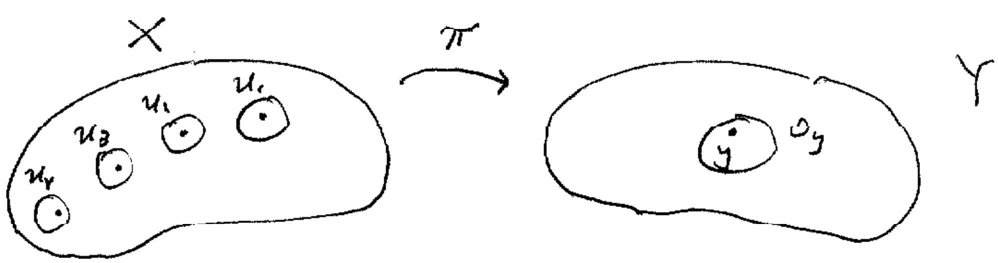
Covering Spaces

It is convenient to assume that topological spaces X, Y are connected "in the cohomological sense" i.e. every locally constant real function on X is globally constant. With this assumption, an epimorphism $X \xrightarrow{\pi} Y$ is called a <u>covering</u> if

for any $y \in Y$, there is open $\mathcal{O}_y \ni y$ such that

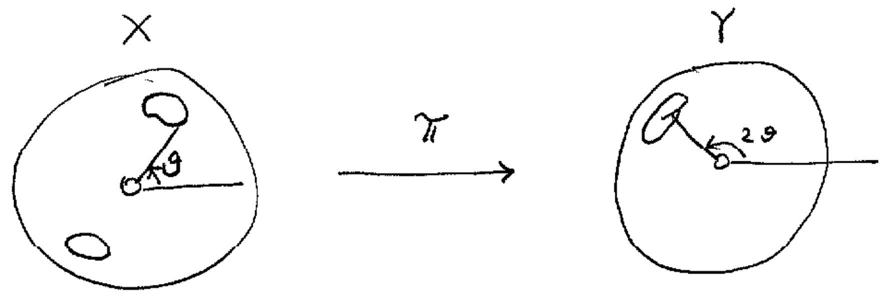
$$\pi^{-1}[\mathcal{O}_y] = \bigcup_i \mathcal{U}_i$$

where $\mathcal{U}_i \subset X$ are disjoint open sets and $\pi : \mathcal{U}_i \to \mathcal{O}_y$ are isomorphisms.



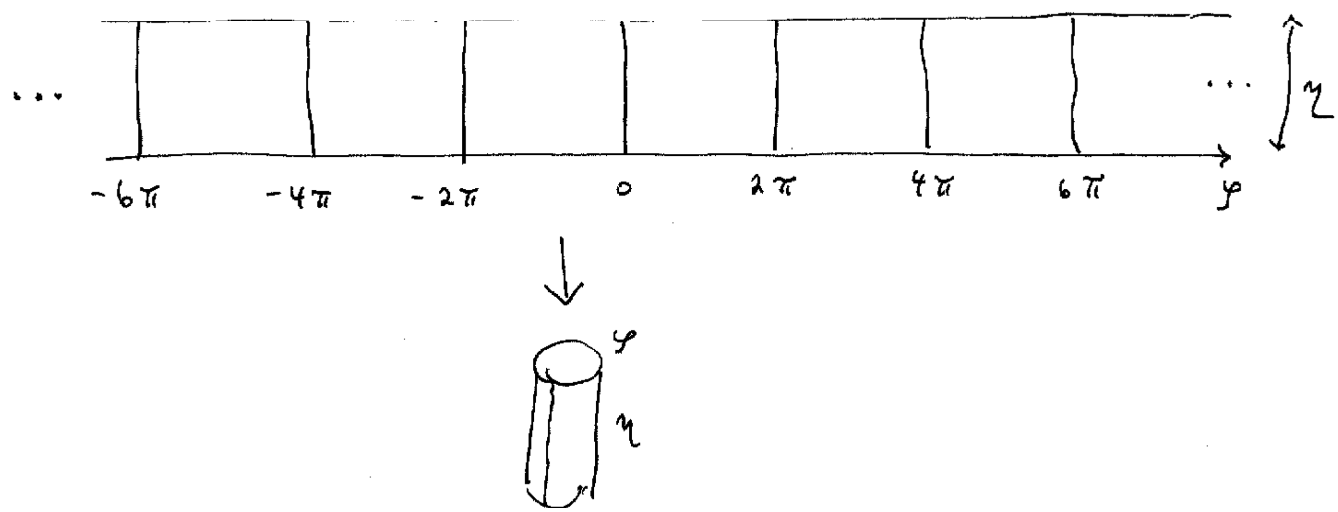
Lemma: degree : $y \mapsto$ # of $\mathcal{U}_1, \dots, \mathcal{U}_n$ neighborhoods is constant on Y. Proof: degree is constant on a neighborhood of any $y \in Y \Rightarrow$ it is globally constant.

Examples



$\pi : (\cos\theta, \sin\theta) \longmapsto (\cos 2\theta, \sin 2\theta)$

is a double covering.



is a countably infinite covering. $O(3) \xrightarrow{\pi} SO(3)$ is a double covering.

Covering spaces is our first example of a fiber bundle. There is a nice simple theory of covering spaces (see, e.g. "Galois' Dream") which we won't ~~had~~ have time to go into e.g.

- For each X, there is a unique "Universal" connected & simply connected covering space.
- Loops in Y can be "lifted" to loops in X causing interesting relationships between homotopy groups.

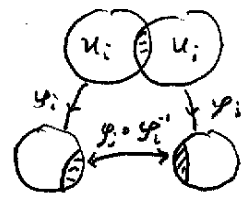
Manifolds

A manifold is a Hausdorff topological space M together with a collection of charts $(\mathcal{U}_i, \varphi_i)$ where

a) $\mathcal{U}_i$ are an open cover of M

b) $\varphi_i : \mathcal{U}_i \to \mathcal{O}_i$ are isomorphisms to open $\mathcal{O}_i \subset \mathbb{R}^n$

c) $\varphi_j \circ \varphi_i^{-1}$ are smooth isomorphisms from $\varphi_i[\mathcal{U}_i \cap \mathcal{U}_j]$ to $\varphi_j[\mathcal{U}_i \cap \mathcal{U}_j]$.



The collection of charts is called an atlas and the $\varphi_j \circ \varphi_i^{-1}$ are sometimes called "transition functions". "smooth" means that all derivatives exist and are continuous.

Manifold is the minimal additional structure needed for doing calculus on a topological space.

Examples of manifolds:

$\mathbb{R}^n$, S^n (n-sphere)

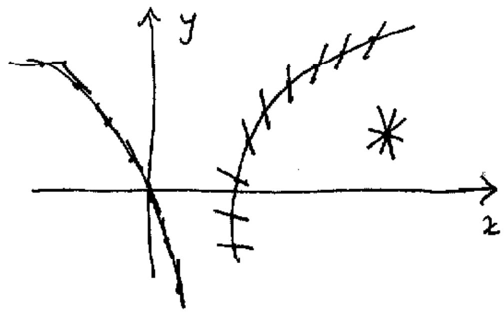
Straight lines through the origin in $\mathbb{R}^3$ ("projective space")

Great circles on a sphere

$SU(2)$, $GL(n)$ etc. (Lie groups).

$C\mathbb{R}^2$: the set of pairs (x, p) where $x \in \mathbb{R}^2$ and p is a line in $\mathbb{R}^2$ through x. "Contact manifold"

Phase space in classical mechanics



Curves in $\mathbb{C}\mathbb{R}^2$

Example. Consider $\mathbb{R}^3 - \{(0,0,0)\}$ with $(x,y,z)\, E\, (x',y',z')$ iff $(x,y,z) = a(x',y',z')$ for some $a \in \mathbb{R}$. We can cover $(\mathbb{R}^3 - \{(0,0,0)\})/E$ with three charts

$$\mathcal{U}_x \ni [(x,y,z)] \xrightarrow{\varphi_x} (y/x,\ z/x)$$
$$\mathcal{U}_y \ni [(x,y,z)] \xrightarrow{\varphi_y} (x/y,\ z/y)$$
$$\mathcal{U}_z \ni [(x,y,z)] \xrightarrow{\varphi_z} (x/z,\ x/z)$$

$$\mathcal{U}_x = \{[(x,y,z)] \text{ with } x \neq 0\}$$
$$\mathcal{U}_y = \{[(x,y,z)] \text{ with } y \neq 0\}$$
$$\mathcal{U}_z = \{[(x,y,z)] \text{ with } z \neq 0\}$$

$\{(\mathcal{U}_x, \varphi_x), (\mathcal{U}_y, \varphi_y), (\mathcal{U}_z, \varphi_z)\}$ are an atlas.

exercise: Fill in the details proving that this is a manifold.

As in other categories, we will typically obtain manifolds by means other than applying the definition directly.

Differential forms in $\mathbb{R}^n$

Remember $\Lambda^k(\mathbb{R}^n) \equiv \underbrace{(\mathbb{R}^n)^* \wedge (\mathbb{R}^n)^* \wedge \cdots \wedge (\mathbb{R}^n)^*}_{k \text{ times.}}$

We define

$$dx_j = ((x_1, x_2, \ldots, x_j, \ldots, x_n) \mapsto x_j) \in (\mathbb{R}^n)^*$$

You can easily see that

$$\{ dx_j \wedge dx_k \wedge dx_l : j < k < l \}$$

is a basis for $\Lambda^k(\mathbb{R}^n)$. The general element of $\Lambda^k(\mathbb{R}^n)$ is $\underbrace{\sum' f_{(i_1, i_2, \ldots, i_k)}}_{\text{differentiable}} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \equiv \sum_I f_I \, dx_I$

These are called "differential forms".

Gradient, differential, divergence, curl are all replaced by the "exterior derivative" defined by

$$d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)$$

a) If $f \in \Lambda^0(\mathbb{R}^n)$, $df = df$ (ordinary differential)

b) $d\left(\sum' f_I \, dx_I \right) = \sum' df_I \, dx_I$

e.g. $\omega = x\,dy + y\,dy$

$$d\omega = dx \wedge dy + dy \wedge dy = dx \wedge dy$$

Where we're headed:

ordinary differential
gradient
divergence
curl

$\Bigg\} \to d$ "exterior derivative"

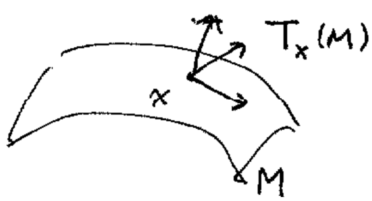
Fundamental theorem of calculus
Gauss Law
Green's theorem
"Stokes" theorem

$\Bigg\}$ Stokes theorem

$$\int_M d\omega = \int_{\partial M} \omega$$

Differential forms on $\mathbb{R}^n \to$ n-Manifolds via a functor.

Each point x in an n-manifold M has an associated vector space



$T_x(M)$ the "tangent space"

$(T_x(M))^*$ the "cotangent space"

Lie Groups: Manifold + Group

$$TM = \bigcup_{x \in M} T_x(M)$$ is called the "tangent bundle"

$TM \xrightarrow{\pi} M$ is the projection epimorphism as in covering spaces

$E \xrightarrow{\pi} M$ in general is a fiber bundle.

Just as in the case of covering spaces, a "lift" of a path from the base space is not unique. In fiber bundles, this lift is specified by a "connection" e.g. A_μ in gauge theories.