Homework # 4 solutions. (S.Y. Feb 2000)

1. Suppose that A_λ are subsets of X. $x \in (\bigcup_\lambda A_\lambda)^c \Leftrightarrow x \notin \bigcup_\lambda A_\lambda$
$\Leftrightarrow x \in A_\lambda$ for all $\lambda \Leftrightarrow x \in A_\lambda^c$ for all $\lambda \Leftrightarrow x \in \bigcap_\lambda A_\lambda^c \Rightarrow (\bigcup_\lambda A_\lambda)^c = \bigcap_\lambda A_\lambda^c$.

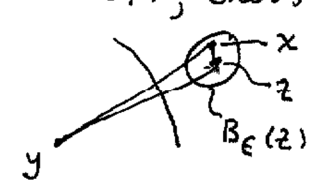
2. Yes. $y \in \varphi[A] \Rightarrow \varphi(x) = y$ for some $x \in A \subset A' \Rightarrow y \in \varphi[A']$.
Yes also, since $x \in \varphi^{-1}[B] \Rightarrow \varphi(x) = y \in B \subset B' \Rightarrow x \in \varphi^{-1}[B']$.

3. By definition, any x in open $\mathcal{O} \subset \mathbb{R}$ contains some open interval $\mathcal{O}_x = (x-\epsilon, x+\epsilon)$ for some $\epsilon > 0$. Then $\mathcal{O} = \bigcup_x \mathcal{O}_x$.

4. Any $A \subset \mathbb{R}$, $A = \bigcup_{x \in A} \{x\} = \bigcup_{x \in A} (\{x\}^c)^c = \bigcap_{x \in A} \{x\}^c$.

5. Check that the open sets form a topology. i) V and $\emptyset$ are obviously open. ii) If $\mathcal{O}_\lambda$ are open, then for any $x \in \mathcal{O}_\lambda$, $v \in V$, there are $\epsilon_\lambda > 0$ s.t. $|a| < \epsilon_\lambda \Rightarrow x + av \in \mathcal{O}_\lambda$. For any $x \in \bigcup_\lambda \mathcal{O}_\lambda$, choose $\mathcal{O}_\lambda \ni x$, $v \in V$, $|a| < \epsilon_\lambda \Rightarrow x + av \in \mathcal{O}_\lambda \subset \bigcup_\lambda \mathcal{O}_\lambda \Rightarrow \bigcup_\lambda \mathcal{O}_\lambda$ is open.
iii) Suppose $\mathcal{O}$ and $\mathcal{O}'$ are open. Then $|a| < \epsilon \Rightarrow x + av \in \mathcal{O}$ and $|a| < \epsilon' \Rightarrow x + av \in \mathcal{O}'$ for $x \in \mathcal{O} \cap \mathcal{O}'$, $v \in V$, $\epsilon, \epsilon' > 0$.
$\Rightarrow |a| < \min(\epsilon, \epsilon') \Rightarrow x + av \in \mathcal{O} \cap \mathcal{O}' \Rightarrow \mathcal{O} \cap \mathcal{O}'$ is open.

6. It's sufficient to show that $\varphi^{-1}[(a,b)]$ is open.
$\varphi^{-1}[(a,b)] = \{x \in X : d(x,y) < b\} \cap \{x \in X : d(x,y) > a\}$.
Since $\{x \in X : d(x,y) < b\}$ is obviously open, we want to show that $A = \{x \in X : d(x,y) > a\}$ is open also. Given $z \in A$, choose $\epsilon = d(z,y) - a > 0$ and let $x \in B_\epsilon(z)$.



$$d(z,y) \leq d(y,x) + d(x,z)$$

$d(x,y) \geq d(z,y) - d(x,z) > d(z,y) - \epsilon = a \Rightarrow x \in A$
$\Rightarrow A$ is open $\Rightarrow \varphi$ is continuous.

7. Notice that φ is continuous due to the direct product

$$\begin{tikzcd} \mathbb{R} & \mathbb{R}\times\mathbb{R} \arrow[l] \arrow[r] & \mathbb{R} \\ & \mathbb{R} \arrow[u, "\varphi"] \arrow[ul, bend left, "\cos"] \arrow[ur, bend right, "\sin"'] & \end{tikzcd} \qquad \varphi : r \mapsto (\cos r, \sin r)$$

The open sets in A are $A \cap \mathcal{O}$ for $\mathcal{O}$ open in $\mathbb{R}^2$.

$\varphi^{-1}[A \cap \mathcal{O}] = \varphi^{-1}[A] \cap \varphi^{-1}[\mathcal{O}] = \varphi^{-1}[\mathcal{O}]$ is open in $\mathbb{R}$.

8. Given $X \xrightarrow{\varphi} Y$, let $x E x'$ iff $\varphi(x) = \varphi(x')$. From the group theory handout, we know that

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & X/E \arrow[d, "\bar{\varphi}"] \\ & Y \end{tikzcd}$$

$\alpha : x \mapsto [x]$

Commutes for unique $\bar{\varphi} : [x] \mapsto \varphi(x)$

The only question is if $\bar{\varphi}$ is continuous.

Applying the powerset / pullback cofunctor,

$$\begin{tikzcd} \mathcal{P}(X) & \mathcal{P}(X/E) \arrow[l, "\alpha^{-1}"'] \\ & \mathcal{P}(Y) \arrow[u, "\bar{\varphi}^{-1}"'] \arrow[ul, "\varphi^{-1}"] \end{tikzcd}$$

Commutes. Given open $\mathcal{O} \in \mathcal{P}(Y)$, $\bar{\varphi}^{-1}[\mathcal{O}]$ has the property that $\alpha^{-1}[\bar{\varphi}^{-1}[\mathcal{O}]] = \varphi^{-1}[\mathcal{O}]$ is open and therefore $\bar{\varphi}^{-1}[\mathcal{O}]$ is open.

9. $x \mapsto \tan(\pi(x - 1/2))$ is an isomorphism from $(0,1)$ to $\mathbb{R}$.

Any isomorphism $\varphi : [0,1] \to \mathbb{R}$ has to be onto $\Rightarrow \varphi[[0,1]] = \mathbb{R}$

$\Rightarrow \Leftarrow$ since $[0,1]$ is compact but $\mathbb{R}$ is not compact.